Supplementary Materials: Demonstration of controlled high-dimensional quantum teleportation

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I. Protocol for teleporting qutrits in symmetric CQT scheme

Suppose Alice has an unknown qutrit $|\psi\rangle=t=\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$ to be teleported, $\alpha, \beta, \gamma$ are complex coefficients satisfying $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$. Initially, sender Alice, receiver Bob, and controller Charlie share a maximally entangled state $(GHZ)_{abc} = \frac{1}{\sqrt{3}} (|000\rangle + |111\rangle + |222\rangle)$. The initial state of the whole system can be rewritten in terms of nine orthogonal 3-dimensional Bell states

$$
|\psi\rangle \otimes (GHZ)_{abc} = \frac{1}{3} \left[ |\phi_{00}\rangle_{ta} (|00\rangle + \beta|11\rangle + \gamma|22\rangle)_{bc} 
+ |\phi_{10}\rangle_{ta} (|00\rangle + \omega^2 \beta|11\rangle + \omega \gamma|22\rangle)_{bc} 
+ |\phi_{20}\rangle_{ta} (|00\rangle + \omega \beta|11\rangle + \omega^2 \gamma|22\rangle)_{bc} 
+ |\phi_{01}\rangle_{ta} (|11\rangle + \beta|22\rangle + \gamma|00\rangle)_{bc} 
+ |\phi_{11}\rangle_{ta} (|11\rangle + \omega^2 \beta|22\rangle + \omega \gamma|00\rangle)_{bc} 
+ |\phi_{21}\rangle_{ta} (|11\rangle + \omega \beta|22\rangle + \omega^2 \gamma|00\rangle)_{bc} 
+ |\phi_{02}\rangle_{ta} (|22\rangle + \beta|00\rangle + \gamma|11\rangle)_{bc} 
+ |\phi_{12}\rangle_{ta} (|22\rangle + \omega^2 \beta|00\rangle + \omega \gamma|11\rangle)_{bc} 
+ |\phi_{22}\rangle_{ta} (|22\rangle + \omega \beta|00\rangle + \omega^2 \gamma|11\rangle)_{bc} \right].
$$

(S1)

where $\omega = e^{2\pi i/3}$, and $|\phi_{jk}\rangle = \frac{1}{\sqrt{3}} \sum_{l=0}^{2} \omega^l |l\rangle |l + k\rangle$ ($l + k$ must be taken from the modula 3) are the nine Bell bases of 3-dimensional Hilbert space. Without loss of generality, we assume that Alice’s measurement is $|\phi_{00}\rangle$, then the state of Bob and Charlie becomes $|\psi\rangle_{bc} = |\phi_{00}\rangle_{bc} + \beta |11\rangle_{bc} + \gamma |22\rangle_{bc}$, which could be rewritten in Charlie’s measurement basis $\{0_x = |0\rangle + |1\rangle + |2\rangle)/\sqrt{3}, |1_x = |0\rangle + \omega |1\rangle + \omega^2 |2\rangle)/\sqrt{3}, |2_x = |0\rangle + \omega^2 |1\rangle + \omega |2\rangle)/\sqrt{3}\}$ as

$$
|\psi\rangle_{bc} = \frac{1}{\sqrt{3}} \left[ (|0\rangle_b + \beta |1\rangle_b + \gamma |2\rangle_b)_{0_x, c} 
+ (|0\rangle_b + \omega^2 \beta |1\rangle_b + \omega \gamma |2\rangle_b)_{1_x, c} 
+ (|0\rangle_b + \omega \beta |1\rangle_b + \omega^2 \gamma |2\rangle_b)_{2_x, c} \right].
$$

(S2)

If controller Charlie allows teleportation from Alice to Bob, he measures qutrit $c$ on the right basis and tells Bob the measurement result. Bob then applies an appropriate 3-dimensional unitary operation to recover state $|\psi\rangle_t$. The
explicit form of the nine 3-dimensional Pauli operators is stated as follows

\[
\begin{align*}
U_{00} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
U_{01} &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
U_{02} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},
U_{10} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},
U_{11} &= \begin{pmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & 0 & \omega \end{pmatrix},
U_{12} &= \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix},
U_{20} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix},
U_{21} &= \begin{pmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
U_{22} &= \begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

(S3)

However, if controller Charlie is unwilling to disclose his measurement results, Bob’s state is a mixed state \( \rho_b = Tr_c(\psi_{bc}(\psi)) \) even when Alice’s results are given. In that case, fidelity between Bob’s final state and Alice’s original state is \( F = Tr(\rho_b|\psi_{r}(\psi)|^2) = |\alpha|^4 + |\beta|^4 + |\gamma|^4 \) and control power is \( P = 1 - F = 1/2 \) over the whole three-dimensional Hilbert space.

II. Control power and bipartite Bell nonlocality for asymmetric CQT scheme

The scheme of controlled teleporting a qubit through a 3-dimensional GHZ channel can be phrased as follows. The information Alice wishes to transmit is encrypted as a qubit \( |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \). And the quantum channel distributed to the three parties is the standard GHZ state \( |GHZ\rangle_{abc} = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle) \). Before transmission, Alice performs a local 3-dimensional Fourier transform \( U \) on the qubit. The matrix form of \( U \) is

\[
U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.
\]

(S4)

Thus the quantum state actually to be teleported is

\[
|\varphi\rangle_r = \frac{\alpha}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle) + \frac{\beta}{\sqrt{3}}(|0\rangle + \omega|1\rangle + \omega^2|2\rangle).
\]

(S5)

To transmit the information, Alice executes a 3×3 Bell-state measurement on qutrits \( ta \). The state of the whole system composed of four qutrits \( abct \) is

\[
|\varphi\rangle_r \otimes |GHZ\rangle_{abc} = \frac{1}{3\sqrt{3}} \left[ (\phi_{00})_{ta} \left( \alpha(|00\rangle + |11\rangle + |22\rangle)_{bc} + \beta(|00\rangle + \omega|11\rangle + \omega^2|22\rangle)_{bc} \right) + (\phi_{10})_{ta} \left( \alpha(|10\rangle + \omega|11\rangle + \omega^2|22\rangle)_{bc} + \beta(|10\rangle + \omega|11\rangle + \omega^2|22\rangle)_{bc} \right) +$$
(\phi_{01})_{ta} \left( \alpha(|00\rangle + \omega^2|11\rangle + \omega|22\rangle)_{bc} + \beta(|00\rangle + |11\rangle + |22\rangle)_{bc} \right) + (\phi_{11})_{ta} \left( \alpha(|00\rangle + \omega^2|11\rangle + \omega|22\rangle)_{bc} + \beta(|00\rangle + |11\rangle + |22\rangle)_{bc} \right) + (\phi_{21})_{ta} \left( \alpha(|00\rangle + |11\rangle + \omega^2|22\rangle)_{bc} + \beta(|00\rangle + |11\rangle + |22\rangle)_{bc} \right) +$$
(\phi_{12})_{ta} \left( \alpha(|00\rangle + |11\rangle + |22\rangle)_{bc} + \beta(|00\rangle + |11\rangle + |22\rangle)_{bc} \right) + (\phi_{22})_{ta} \left( \alpha(|00\rangle + |11\rangle + |22\rangle)_{bc} + \beta(|00\rangle + |11\rangle + |22\rangle)_{bc} \right) \right].
\]

(S6)

For instance, the state of 2-qutrit system \( ta \) is projected onto \( |\phi_{00}\rangle_{ta} \), then the state of 2-qutrit system \( bc \) becomes \( |\psi\rangle_{bc} = \alpha|00\rangle_{bc} + \beta|10\rangle_{bc} \). Suppose Charlie allows the success of the teleportation, he would measure qutrit \( c \) in the 3-dimensional x-basis \( \left\{ |\hat{x}_0\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle), |\hat{x}_1\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \omega|1\rangle + \omega^2|2\rangle), |\hat{x}_2\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \omega^2|1\rangle + \omega|2\rangle) \right\} \). If the measurement outcome is \( |\hat{x}_0\rangle \), the state of qutrit \( b \) would be correspondingly projected onto \( |\psi\rangle_{b} = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle) \). Then Bob can perform an additional local operation \( U^{-1} \) on qutrit \( b \) to successfully achieve the transmitted information as \( |\psi\rangle_{b}^{\text{allow}} = \alpha|0\rangle + \beta|1\rangle \). For any other combinations of local measurement
outcomes from Alice and Charlie, there is always a local unitary operation strategy for Bob to recover the initial state $|\psi\rangle$. Consequently, the teleportation with Charlie’s full permission is $F_{\text{allow}} = 1$. Suppose Charlie doesn’t allow the success of teleportation, he would refuse to measure qutrit $c$ and randomly published measurement results, thus the state of qutrit $b$ is a mixed state as $\rho_b^{\text{deny}} = Tr_a(|\psi\rangle_{bc} \langle \psi|)$. In this situation, Bob may execute local $U^{-1}$ to recover as much as information, and the final state of qutrit $b$ becomes

$$\rho_b^{\text{deny}} = \frac{1}{3}(\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle) + \frac{1}{3}(\alpha|2\rangle + \beta|0\rangle)(\alpha|2\rangle + \beta|0\rangle) + \frac{1}{3}(\alpha|1\rangle + \beta|2\rangle)(\alpha|1\rangle + \beta|2\rangle).$$  

(S7)

In this case, the average teleportation fidelity is calculated as

$$F_{\text{deny}} = \frac{I \text{Tr}(\rho_b^{\text{deny}} |\psi\rangle \langle \psi| d\alpha d\beta)}{I d\alpha d\beta} = \int \left( \frac{1}{3} + \frac{2}{3}|\beta|^2 \right) d\alpha d\beta = \frac{4}{9}. \quad \text{(S8)}$$

Charlie is able to control the teleportation between Alice and Bob by changing the measurements implemented. In our experimental case, Charlie implemented $M_0$ measurement with a probability of $p$, while he did not measure his photon and randomly published measurement results with a probability of $(1 - p)$. The average fidelity of teleporting a qubit through a 3-dimensional GHZ channel under Charlie’s control is

$$F_{2,3} = 4 + 5p. \quad \text{(S9)}$$

The control power $P$ of CQT is defined as Charlie’s control range of the average teleportation, and in this asymmetric CQT case, the theoretical value of control power is $P_{2,3} = \frac{2}{3}$. This value can be demonstrated by preparing teleportation states $|\psi_1\rangle = |\bar{0}\rangle$, $|\psi_2\rangle = |\bar{1}\rangle$, $|\psi_3\rangle = (|\bar{0}\rangle + |\bar{1}\rangle)/\sqrt{2}$, $|\psi_4\rangle = (|\bar{0}\rangle - |\bar{1}\rangle)/\sqrt{2}$, $|\psi_5\rangle = (|\bar{0}\rangle + i|\bar{1}\rangle)/\sqrt{2}$, $|\psi_6\rangle = (|\bar{0}\rangle - i|\bar{1}\rangle)/\sqrt{2}$ from 3 mutually unbiased bases for a qubit system in the optical experiment described in the main text, and the results show the average fidelity of $F_{\text{allow}} = 97.1 \pm 0.3\%$ when $p = 1$ and $F_{\text{deny}} = 44.3 \pm 0.3\%$ when $p = 0$. Thus the measured value of control power is $F_{2,3} = 52.8 \pm 0.3\%$, which is well consistent with the theoretical results. Compared with the symmetric 2-dimensional CQT scheme wherein the control power is $P_{2,2} = \frac{1}{2}$, it is apparent to see a significant improvement in Charlie’s control power, and the basic reason for this improvement consists in a larger control range over the bipartite Bell nonlocality $[1]$ between Alice and Bob in high-dimensional entangled state channels.

To explain the improvement of control power in detail, we will analyze the channel structure and extract the parts that actually use the asymmetric CQT protocol. In the asymmetric CQT, before Charlie makes the decision of whether to measure his qutrit $c$, the whole state of 4-qutrit system $\tilde{t}_{abc}$ is

$$|\psi\rangle_{tabc} = (M_{ta} \otimes U_b^{-1}) U_t |\psi\rangle_{t(GHZ)}_{abc}. \quad \text{(S10)}$$

Here, $M_{ta} = \sum_{j,k=0}^{2} |\phi_{jk}\rangle_{ta} \langle \phi_{jk}|$ is the Bell-state-analysis operator. In our experiment, qutrits $t$ and $a$ are chosen to be projected onto $|\phi_{00}\rangle_{ta}$ with probability $p_{00} = \frac{1}{2}$. For convenience, one can only consider the situation of $jk = 00$, and $|\psi\rangle_{tabc}$ becomes

$$|\psi_{00}\rangle_{tabc} = |\phi_{00}\rangle_{ta} \langle \phi_{00}| (U_t \otimes I_a \otimes U_b^{-1} \otimes I_c) |\psi\rangle_{t(GHZ)}_{abc}. \quad \text{(S11)}$$

As $(U_t \otimes I_a) |\phi_{00}\rangle_{ta} = (I_t \otimes U_a) \langle \phi_{00}|_{ta}$, Eq. (S11) can be rewritten as

$$|\psi_{00}\rangle_{tabc} = |\phi_{00}\rangle_{ta} \langle \phi_{00}| \left[ |\psi\rangle_{t} \otimes (U_a \otimes U_b^{-1} \otimes I_c \langle GZH\rangle_{abc}) \right]. \quad \text{(S12)}$$

From Eq. (S12), one can see that local operations and Bell-state-analysis measurement can transform the channel state $|GZH\rangle_{abc}$ into an equivalent one as $|\tilde{\psi}_{abc}\rangle = (U_a \otimes U_b^{-1} \otimes I_c) |GZH\rangle_{abc}$, and its form is

$$|\tilde{\psi}_{abc}\rangle = \frac{1}{3} \left[ (|0\rangle + |1\rangle) |1\rangle + |2\rangle |2\rangle \rangle_{ab} |0_x\rangle_c \\
+ (|0\rangle |0\rangle + |1\rangle |1\rangle + |2\rangle |2\rangle \rangle_{ab} |1_x\rangle_c \\
+ (|0\rangle |0\rangle + |1\rangle |1\rangle + |2\rangle |2\rangle \rangle_{ab} |2_x\rangle_c \right]. \quad \text{(S13)}$$

In the 3-qutrit entangled state $|\tilde{\psi}_{abc}\rangle$, the subspace used to transmit a qubit information is

$$|\tilde{\psi}_{abc}\rangle = \frac{1}{\sqrt{6}} \left[ (|0\rangle + |1\rangle) |1\rangle \rangle_{ab} |0_x\rangle_c \\
+ (|0\rangle |2\rangle + |1\rangle |0\rangle) \rangle_{ab} |1_x\rangle_c \\
+ (|0\rangle |1\rangle + |1\rangle |2\rangle \rangle_{ab} |2_x\rangle_c \right]. \quad \text{(S14)}$$
From Eq. (S14), one can see that when Charlie fully allows the teleportation between Alice and Bob, the channel can be projected onto 2-dimensional maximally entangled state, such as $|\psi\rangle_{ab}^{allow} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, by Charlie’s measurement. When Charlie controls the teleportation, the channel state between Alice and Bob is projected onto a mixed state, and Bob will randomly exploit two of the three dimensions of qutrit $b$ to recover the initial state $|\psi\rangle$. For example, if Bob exploits the subspace $\{|0\rangle, |1\rangle\}$ of channel qutrit $b$ to extract the transmitted information, the bipartite quantum channel between Alice and Bob controlled by Charlie’s measurement is

$$\rho_{ab} = \frac{p}{2}(|00\rangle\langle00|) + (1 - p)Tr_c(|\tilde{\psi}^{'}\rangle_{abc}\langle\tilde{\psi}^{'}|)_{bc\{00,11\}} + \frac{1 - p}{6} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$  

(S15)

The performance of teleportation is generally thought to be linked to the quantum correlation of the channel. One remarkable progress is Horodecki’s theorem [5] on the relationship between the fidelity of teleporting a qubit and the corresponding bipartite Bell-nonlocality of the used channel, which is represented by the maximal violation of the CHSH inequality.

Following [6], the maximal violation of the CHSH inequality for channel Eq. (S15) is given by $B_{asy} = 2\sqrt{2(\frac{\sqrt{1 + 2p^3}}{3})^2}$, and the teleportation fidelity of this asymmetric case is $F_{asy} = \frac{4 + 5p}{9}$. For the symmetric case (teleporting a qubit through a 2-dimensional GHZ channel), the maximal violation of the CHSH inequality and average teleportation fidelity is $B_{sym} = 2\sqrt{1 + p^2}$ and $F_{sym} = \frac{2 + p}{6}$ respectively. That is, the asymmetric scheme has a larger control range and higher control power than the symmetric scheme. When transmitting one qubit information, both asymmetric and symmetric scheme has a fidelity of 100% with the controller’s allowance, while the asymmetric scheme has a lower fidelity than the symmetric scheme if the controller is unwilling to cooperate. As shown in Fig. S1, compared with the symmetric 2-dimensional CQT scheme, the asymmetric CQT scheme utilizing high-dimensional entangled resources exhibits a stronger manipulation range of the Bell nonlocality between Alice and Bob, thus leading to a larger control range of teleportation fidelity while transmitting the same information. As a consequence, the control power of the asymmetric CQT protocol is higher than the symmetric one. This result shows a prominent application of high-dimensional asymmetric CQT schemes, such as a reliable quantum network with stronger control and supervision ability.

FIG. S1. (a) Theoretical results of the bipartite CHSH value between Alice and Bob under the control of Charlie: $B_{asy} = \sqrt{2(\frac{\sqrt{1 + 2p^3}}{3})^2}$ (red solid curve, the asymmetric case) and $B_{sym} = 2\sqrt{1 + p^2}$ (black dashed curve, the symmetric case). (b) Theoretical results of the average teleportation fidelity under the control of Charlie: $F_{asy} = \frac{4 + 5p}{9}$ (red solid curve, the asymmetric case) and $F_{sym} = \frac{2 + p}{6}$ (black dashed curve, the symmetric case).
As shown in Fig. S2 we encode photons in path a1 as $|0\rangle$, photons in path a2 as $|1\rangle$, and photons in path a3 as $|2\rangle$. Furthermore, vertically polarized photons passing through a11, a21 and a31 are encoded as $|0\rangle$, horizontally polarized photons passing through a12, a22 and a32 are encoded as $|1\rangle$ and vertically polarized photons passing through a12, a22 and a32 are encoded as $|2\rangle$. Thus, by setting QH1 and QH4 (QH: the combination of a QWP and an HWP) properly we transform $|0\rangle_a$ to $|0\rangle_a \otimes |t\rangle$, while $|t\rangle = \alpha |0\rangle + \beta |1\rangle + \gamma |2\rangle$ is the state to be teleported, $\alpha, \beta, \gamma$ are complex coefficients. We can generate arbitrary $(1/\sqrt{3})(|000\rangle_{abc} + |111\rangle_{abc} + |222\rangle_{abc}) \otimes |t\rangle$ by similarly tuning QH1, QH2, QH3 and QH4, QH5, QH6, the degrees of which are shown in Table S1.

We assume that Alice’s measurement is $B_{00}$ without loss of generality. If not, an appropriate three-dimensional unitary operation is applied. To distinguish $\{B_{00}, I - B_{00}\}$, we pick vertically polarized photons in path a11 ($|00\rangle$), horizontally polarized photons in path a22 ($|11\rangle$) and vertically polarized photons in path a32 ($|22\rangle$) by setting the angles of the HWPs shown in Fig. S3 while blocking the others, a12, a21 and a31. In this way, one of nine Bell bases of three-dimensional Hilbert space $B_{00}$ is detected, leaving others undetected.
FIG. S3. Partial Bell state measurement. In order to distinguish \{B_{00}, I - B_{00}\}, we pick vertically polarized photons in path a11, horizontally polarized photons in path a22, and vertically polarized photons in path a32 by setting the angles of HWPs as shown in the figure, and block all the photons in the other paths.

IV. Experimental implementation of Charlie’s measurement and Bob’s measurement

After Alice’s state preparation and Bell state measurement, the state of Bob and Charlie becomes |\phi⟩_{bc} = \alpha |00⟩_{bc} + \beta |11⟩_{bc} + \gamma |22⟩_{bc}. Controller Charlie chooses a three-dimensional complete orthogonal measurement basis \{M_0 = (|0⟩ + |1⟩ + |2⟩)/\sqrt{3}, M_1 = (|0⟩ + \omega |1⟩ + \omega^2 |2⟩)/\sqrt{3}, M_2 = (|0⟩ + \omega^2 |1⟩ + \omega |2⟩)/\sqrt{3}\} shown in Fig. S4 and Table S2 to measure the qutrit and tells Bob the measurement result.

Then Bob implements a standard quantum state tomography for state reconstruction, comparing it with Alice’s original state. Two BDs, HWPs, a QWP, and a PBS are used to realize state tomography of the spatial mode shown in Fig. S5 and Table S3.

FIG. S4. Charlie’s measurement. By setting QWP1, HWP1, HWP2, QWP3, and HWP3 properly, Charlie performs a three-dimensional complete orthogonal measurement using the basis \{M_0 = (|0⟩ + |1⟩ + |2⟩)/\sqrt{3}, M_1 = (|0⟩ + \omega |1⟩ + \omega^2 |2⟩)/\sqrt{3}, M_2 = (|0⟩ + \omega^2 |1⟩ + \omega |2⟩)/\sqrt{3}\}. 
TABLE S2. Charlie’s measurement. In order to perform a three-dimensional complete orthogonal measurement on basis \( \{ M_0 = (\lvert 0 \rangle + \lvert 1 \rangle + \lvert 2 \rangle)/\sqrt{3}, M_1 = (\lvert 0 \rangle + \omega \lvert 1 \rangle + \omega^2 \lvert 2 \rangle)/\sqrt{3}, M_2 = (\lvert 0 \rangle + \omega^2 \lvert 1 \rangle + \omega \lvert 2 \rangle)/\sqrt{3} \} \), the degrees of QWPs and HWPs are well set as given in the table.

<table>
<thead>
<tr>
<th>QWP1</th>
<th>HWP1</th>
<th>HWP2</th>
<th>QWP3</th>
<th>HWP3</th>
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<td>0º</td>
<td>0º</td>
<td>0º</td>
</tr>
<tr>
<td>( M_1 )</td>
<td>45º</td>
<td>-22.5º</td>
<td>45º</td>
<td>-58.3º</td>
</tr>
<tr>
<td>( M_2 )</td>
<td>45º</td>
<td>37.5º</td>
<td>45º</td>
<td>-58.3º</td>
</tr>
<tr>
<td>( M_3 )</td>
<td>45º</td>
<td>37.5º</td>
<td>45º</td>
<td>-58.3º</td>
</tr>
<tr>
<td>( M_4 )</td>
<td>45º</td>
<td>37.5º</td>
<td>45º</td>
<td>-58.3º</td>
</tr>
<tr>
<td>( M_5 )</td>
<td>45º</td>
<td>37.5º</td>
<td>45º</td>
<td>-58.3º</td>
</tr>
<tr>
<td>( M_6 )</td>
<td>45º</td>
<td>37.5º</td>
<td>45º</td>
<td>-58.3º</td>
</tr>
</tbody>
</table>

FIG. S5. Bob’s measurement. We use BDs, HWPs, a QWP, and a PBS to realize state tomography of the path DOF.

TABLE S3. Bob’s measurement. Detailed degrees of waveplates to realize three-dimensional quantum state tomography for the path DOF are shown above.

<table>
<thead>
<tr>
<th>HWP1</th>
<th>HWP2</th>
<th>HWP3</th>
<th>HWP4</th>
<th>HWP5</th>
<th>HWP6</th>
<th>QWP</th>
</tr>
</thead>
<tbody>
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<td>0º</td>
<td>0º</td>
<td>0º</td>
<td>0º</td>
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<tr>
<td>( \lvert 1 \rangle )</td>
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<td>0º</td>
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<td>0º</td>
<td>0º</td>
</tr>
<tr>
<td>( \lvert 2 \rangle )</td>
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<td>0º</td>
</tr>
<tr>
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<td>0º</td>
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<tr>
<td>( \lvert 0 \rangle + \lvert 2 \rangle )</td>
<td>0º</td>
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<td>45º</td>
<td>0º</td>
</tr>
<tr>
<td>( \lvert 1 \rangle + \lvert 2 \rangle )</td>
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<td>0º</td>
<td>45º</td>
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<td>45º</td>
<td>0º</td>
</tr>
</tbody>
</table>

V. Detailed experimental results

We reconstruct the density matrix of \( \lvert t_1 \rangle - \lvert t_2 \rangle, \lvert \psi_1 \rangle - \lvert \psi_2 \rangle \) by using the standard qutrit state tomography and the results are shown in Table S4 and Table S5.

VI. Noise model

For example, if Alice prepares \( \lvert t_4 \rangle = (\lvert 0 \rangle + \lvert 1 \rangle + \lvert 2 \rangle)/\sqrt{3} \) to teleport and Charlie’s measurement basis is \( M_0 = (\lvert 0 \rangle + \lvert 1 \rangle + \lvert 2 \rangle)/\sqrt{3} \), then Bob’s state will collapse to \( \lvert \psi \rangle = (\lvert 0 \rangle + \lvert 1 \rangle + \lvert 2 \rangle)/\sqrt{3} \) theoretically, that is fidelity \( F = Tr(\rho_{b} \lvert t_4 \rangle \langle t_4 \lvert) = 1 \). Nevertheless, in our experiment, the measured fidelity is lower than the theoretical value mainly due to imperfect interference of different paths and imperfect polarization.

In our experiment, the target bipartite path entangled state is \( \lvert \phi \rangle_{ab} = (\lvert 00 \rangle + \lvert 11 \rangle + \lvert 22 \rangle)/\sqrt{3} \). However, the state we generated is

\[
\rho_{ab} = 3p_1 \lvert \phi \rangle_{ab} \langle \phi \rvert + p_2 (\lvert 00 \rangle \langle 00 \rvert + \lvert 11 \rangle \langle 11 \rvert + \lvert 22 \rangle \langle 22 \rvert) + p_3 (\lvert 01 \rangle \langle 01 \rvert + \lvert 02 \rangle \langle 02 \rvert + \lvert 10 \rangle \langle 10 \rvert + \lvert 12 \rangle \langle 12 \rvert + \lvert 20 \rangle \langle 20 \rvert + \lvert 21 \rangle \langle 21 \rvert),
\]

(S16)
Thus, if Charlie’s measurement basis is $\{+\} \pm \{\pm\}$, after Alice’s state preparation and Bell state measurement, the fidelity is $V = \sin^2 (\theta/2)$ and the polarization interference visibility is $V_{\text{polarization}} = 1 - \cos^2 (\theta/2)$. The path interference visibility of Eq. (S16) is $V_{\text{path}} = \frac{p_1}{p_1 + p_2 + p_3}$ and the polarization interference visibility is $V_{\text{polarization}} = \frac{p_1 + p_2 - p_3}{p_1 + p_2 + p_3}$. In our experiment, the measured path interference visibility is $V_{\text{path}} = 0.961$ and polarization interference visibility is $V_{\text{polarization}} = 0.990$, so the fidelity of Bob’s state is $F = Tr (\rho_b (|t_4\rangle \langle t_4|)) = 0.971$ via calculation, which fits well with experimental results.

VII. Transformation of linear optical devices

In Fig. ??, we mention some linear optical devices, such as quarter-wave plate, half-wave plate, and polarizing beam splitter. Here we introduce the transformation matrices of these linear optical devices.
We denote horizontally polarized photon state as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and vertically polarized photon state as $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then the transformation matrix of PBS is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, the transformation matrix of HWP is $\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$, and the transformation matrix of QWP is $\frac{\begin{pmatrix} 1 - \text{i}\cos(2\theta) & -\text{i}\sin(2\theta) \\ -\text{i}\sin(2\theta) & 1 + \text{i}\cos(2\theta) \end{pmatrix}}{\sqrt{2}}$, where $\theta$ is rotation angle.

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