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The GM(1,1) Model Optimized by Using Translation Transformation Method and Its Application of Rural Residents' Consumption in China

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Abstract Conventional GM(1,1) model shows some limitations which affect directly to the model applicability as well as prediction accuracy. In order to improve the disadvantages existing in the original grey model, the authors analyze the translation transformation approach to optimize the raw sequence for grey modeling. This paper mainly carries on the proof of the existence of translation transformation from the perspective of smooth ratio and gives the method to solve the best step size of translation transformation. Then, the authors applied the grey models optimized by translation transformation to analyze the problem of the consumption of rural residents in China. As a result, the authors find GM(1,1) model optimized by translation transformation has characteristics of higher accuracy and simple calculation process and an important practical application in residents consumption.

Keywords grey theory; translation transformation; GM (1,1) model; consumption

1 Introduction

The grey model GM(1,1) has been developed and successfully applied to many fields in modern life such as social sciences, agriculture, traffic, transportation, marketing, business, engineering sciences etc. and achieved promising results. But as same as other forecasting models, the grey model GM(1,1) has some limitations. It is only applied to single exponential growth sequence, and the prediction results can achieve good accuracy. On the other hand, we can also know from the basic form of the grey prediction model that the GM(1,1) model is constructed according to the linear changing relationship between the raw data sequence and its 1-AGO sequence, but the variation usually appears in many systems is not linear between them, so many questions need to improve the GM(1,1) model for a higher prediction model accuracy. At present, there are some improvements for the GM(1,1) model as follows:

- The improvements of estimate parameters of the GM(1,1) model[1, 2].

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- The improvements of the background series[3–5].
- The improvements of the initial condition of the GM(1,1) model[6, 7].
- The improvements of model accuracy from the form of the GM(1,1) model[8–10].

Besides as above, there are a lot of improvements from the form of the GM(1,1) model, such as the GGM model[11] is a simple improvement model, Zhang et al.[12] applied derivatives to construct the grey model, Zhao et al.[13] used a grey model optimized by differential evolution algorithm to forecast the per capita annual net income of rural households in China and Wu et al.[14] presented the improvement model by using of the fractional order accumulation. Considering applicability in the case of limited data situations and requirement of little computational effort, grey prediction with rolling mechanism[15] approach is proposed. When the data sequence changes are not smooth, the SAGM(1,1) model[16] can replace the GM(1,1) model for higher accuracy. All these improvement methods can improve the model accuracy from a certain degree, but there is no research which fully investigates all the limitations of GM(1,1) model to design a new accurate grey prediction model with high adaptability to all application purposes. So it is necessary to find a simple approach for high model accuracy. We found the translation transformation is a very simple method, and it can be better to improve model accuracy. At present, the research on improving the grey model by using of the sequence transformation method[17] is few, thus the studies on translation transformation for improving grey model accuracy is important significance.

The problem of residents' consumption is an important part of the study of economics, and now there are a lot of studies on the residents' consumption by using of regression analysis method[18–21]. Zhang et al.[22] employs the Euler equation approach to examine the effect of monetary policy, conducted through the interest rate channel, on household consumption. Haddad[23] proposed a prediction model for recurrent consumption behaviors based on inhomogeneous Poisson processes aiming at predicting. Because the problem of residents' consumption exist information incompleteness and has a certain characteristic of exponential growth, the grey modeling method has good applicability. Comparing with other models, the grey model has a simple calculation process and high model accuracy. However, the study on residents' consumption by using of grey models is very few. In the paper, we analyzed the residents' consumption by applying grey models which optimized by the best translation transformation method.

2 The Basic Concept of Grey Modeling

2.1 GM(1, 1)model

The steps of GM(1, 1) model are shown as follows.

Step 1 Original time sequence with n samples is defined as

$$X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n)) \tag{1}$$

Step 2 Construct monotonic increasing sequence $X^{(1)}$ by a one-time accumulated generating operation (I-AGO), expressed as

$$X^{(1)} = (x^{(1)}(1), x^{(1)}(2), \cdots, x^{(1)}(n)) \tag{2}$$

where

$$x^{(1)}(k)=\sum_{i=1}^{k}x^{(0)}(i),\quad k=1,2,\cdots,n$$

Step 3 Form the GM(1,1) model by establishing a first order grey differential equation:

$$x^{(0)}(k)+az^{(1)}(k)=b \tag{3}$$

where $z^{(1)}(k)=\frac{1}{2}(x^{(1)}(k)+x^{(1)}(k-1))$. In Equation (3), $k\,(k=2,3,\cdots,n)$ is a time point. a is called the development coefficient and b is called grey action coefficient. Using least mean square estimation technique coefficients, $[a,b]^{\mathrm{T}}$ can be estimated as

$$[a,b]^{\mathrm{T}}=(B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}Y \tag{4}$$

where

$$Y=\begin{bmatrix}x^{(0)}(2)\\x^{(0)}(3)\\\vdots\\x^{(0)}(n)\end{bmatrix},\quad B=\begin{bmatrix}-z^{(1)}(2)&1\\-z^{(1)}(3)&1\\\vdots&\vdots\\z^{(1)}(n)&1\end{bmatrix} \tag{5}$$

Step 4 According to the estimated coefficients a and b, the grey prediction equation can be obtained by solving differential equation in Equation (3).

$$\tilde{x}^{(1)}(k+1)=\left(x^{(1)}(1)-\frac{b}{a}\right)\mathrm{e}^{-ak}+\frac{b}{a} \tag{6}$$

where $\tilde{x}^{(1)}(k)$ denotes the prediction of x at k time point.

Step 5 To obtain the forecasting values of $\tilde{x}^{(0)}(k), k=2,3,\cdots,n$, the inverse accumulated generating operation (I-AGO) is used to establish the following Grey model:

$$\begin{cases}\tilde{x}^{(1)}(k+1)=\left(x^{(1)}(1)-\dfrac{b}{a}\right)\mathrm{e}^{-ak}+\dfrac{b}{a}\\ \tilde{x}^{(0)}(k+1)=\tilde{x}^{(1)}(k+1)-\tilde{x}^{(1)}(k)\end{cases} \tag{7}$$

We can also simplify Equation (7) as

$$\tilde{x}^{(0)}(k+1)=(1-\mathrm{e}^{a})\left(x^{(1)}(1)-\frac{b}{a}\right)\mathrm{e}^{-ak} \tag{8}$$

The advantages of grey prediction model GM(1,1) is easy to calculate and suitable for many problems of approximate exponential change. GGM(1,1) model which has more simple form is more suitable for the approximate exponential sequence and has a simple calculation process, so we compared the improved model GGM(1, 1) with GM(1,1) model applied the translation transformation method in this paper.

2.2 GGM(1, 1) Model[11]

e^{-a} in Equation (8) was replaced with ϖ as follows:

$$\tilde{x}^{(0)}(k+1)=(1-\varpi^{-1})\left(x^{(1)}(1)-\frac{b}{a}\right)\varpi^{k} \tag{9}$$

We put Equation (9) into $f(\varpi) = \sum_{k=2}^{n}[x^{(0)}(k) - \tilde{x}^{(0)}(k)]^2$ making the function $f(\varpi)$ reach the minimum, then the variable ϖ was obtained. Hence, the GGM (1,1) model could be expressed as follows:

$$\begin{cases} f(\varpi) = \min \sum_{k=2}^{n}[x^{(0)}(k) - \tilde{x}^{(0)}(k)]^2 \\ \tilde{x}^{(0)}(k+1) = (1 - \varpi^{-1})\left(x^{(1)}(1) - \dfrac{b}{a}\right)\varpi^k \end{cases} \tag{10}$$

where, the problem of solving the function $f(\varpi)$ belongs to the unconstrained nonlinear continuously differentiable mathematical programming, which could be solved by the GA method. The GGM (1,1) model has a higher accuracy comparing with the GM (1,1) model in theory. However, the GGM (1,1) model has some limitations as well as the GM (1,1) model is not suitable for a non-smooth data sequence.

2.3 Testing for the Grey Model

If the grey model achieves a certain predictive accuracy, not only the development coefficient a takes a certain value, but also the error of the grey model must pass the error test. The calculation error can be defined as follows.

- The average absolute error

$$\bar{\varepsilon} = \frac{1}{n}\sum_{i=1}^{n}|\tilde{x}^{(0)}(i) - x^{(0)}(i)| \tag{11}$$

- The average relative error

$$\bar{\varepsilon}_r = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{\tilde{x}^{(0)}(i) - x^{(0)}(i)}{x^{(0)}(i)}\right| \tag{12}$$

Generally, if the grey model passes the test that shows in Table 1, it can be applied to forecast. Otherwise, we must improve the grey model for a higher accuracy.

Table 1 The precision reference of the grey model

Accuracy class	$\bar{\varepsilon}_r$	$\lvert a\rvert$	Practicability of the grey model
Class-I	$< 2\%$	< 0.3	Medium or long term
Class-II	$< 10\%$	< 0.5	Short-term
Class-III	$< 20\%$	< 0.8	Short-term used carefully
Class-IV	$\geq 20\%$	≥ 0.8	Not applicable

3 Translation Transformation

3.1 The Existence of Translation Transformation

Definition 1 Assume that a non-negative increasing sequence $X = (x(1), x(2), \cdots, x(n))$ by using of translation transformation is expressed as

$$\bar{X} = (x(1) - \tau, x(2) - \tau, \cdots, x(n) - \tau)$$

where τ is translation step and $\tau > 0$.

Definition 2 The smooth ratio of a non-negative increasing sequence $X = (x(1), x(2), \cdots, x(n))$ is defined as

$$\rho(k) = \frac{x(k)}{\sum_{i=1}^{k-1} x(i)}, \quad k = 2, 3, \cdots, n$$

Obviously, the smooth ratio $\rho(k)$ is smaller, the raw sequence X is more smoothly.

Property 1 *For any non-negative increasing sequence $X = (x(1), x(2), \cdots, x(n))$, existing translation transformation τ can make smooth ratio $\rho(k)$ smaller.*

Proof Assumed that translation transformation τ exists and there is the inequality

$$\rho_\tau(k) < \rho(k) \tag{13}$$

i.e.

$$\frac{x(k) - \tau}{\sum_{i=1}^{k-1}(x(i) - \tau)} < \frac{x(k)}{\sum_{i=1}^{k-1} x(i)}$$

From the inequality as above, we obtained

$$\begin{aligned} &\frac{x(k) - \tau}{\sum_{i=1}^{k-1} x(i) - (k-1)\tau} < \frac{x(k)}{\sum_{i=1}^{k-1} x(i)} \\ \Rightarrow\ &\frac{[(k-1)x(k) - \sum_{i=1}^{k-1} x(i)]\tau}{\sum_{i=1}^{k-1} x(i)[\sum_{i=1}^{k-1}(x(i) - (k-1)\tau)]} < 0 \\ \Rightarrow\ &\frac{\tau}{\sum_{i=1}^{k-1}(x(i) - (k-1)\tau)} < 0 \end{aligned}$$

Then,we obtained $\tau < 0$ or $\tau > \frac{\sum_{i=1}^{k-1}(x(i)}{k-1}$.

Thus, Equation (13) was satisfied, i.e. we could find the translation step τ in the condition assumed as above. The proof has been completed. □

3.2 The Best Translation Step

If the grey models achieve higher prediction accuracy, the translation step selected must be appropriate. The steps of solving the translation step are defined as follows.

Step 1 Let $\bar{x}^{(0)}(k) = x^{(0)}(k) - \tau$, then

$$\bar{x}^{(1)}(k) = \sum_{i=1}^{k} \bar{x}^{(0)}(i)$$

Step 2 To build linear model as follows:

$$\bar{x}^{(0)}(k) = \alpha \bar{x}^{(1)}(k) + \beta$$

Hence

$$x^{(0)}(k) - \tau = \alpha x^{(1)}(k) - \alpha k\tau + \beta \tag{14}$$

Step 3 Let $c_1 = \alpha\tau, c_2 = \beta + \tau$, Equation (14) can be expressed as

$$x^{(0)}(k) = \alpha x^{(1)}(k) - c_1 k + c_2 \tag{15}$$

Step 4 According to the least squares principle, the parameters $[\alpha, c_1, c_2]^{\mathrm{T}}$ is defined as Equation (16).

$$[\alpha, c_1, c_2]^{\mathrm{T}} = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}Y \tag{16}$$

where

$$Y = \begin{bmatrix} x^{(0)}(1) \\ x^{(0)}(2) \\ \vdots \\ x^{(0)}(n) \end{bmatrix}, \quad B = \begin{bmatrix} -z^{(1)}(2) & -1 & 1 \\ -z^{(1)}(3) & -2 & 1 \\ \vdots & \vdots & \\ z^{(1)}(n) & -n & 1 \end{bmatrix}$$

Then, we obtain the translation step $\tau = c_1/\alpha$.

4 The Consumption of Urban Residents in China

The data of the consumption of rural residents in China from 1995 to 2012 as Table 2 was obtained from National Bureau of Statistics, by which we could construct grey models to forecast the consumption of urban residents in China from 2013 to 2016.

Table 2 The consumption of urban residents in China (2001–2012)

Year	Consumption (yuan)	Year	Consumption (yuan)	Year	Consumption (yuan)
1995	1310.4	2001	1741.1	2007	3223.9
1996	1572.1	2002	1834.3	2008	3660.7
1997	1617.2	2003	1943.3	2009	3993.5
1998	1590.3	2004	2184.7	2010	4381.8
1999	1577.4	2005	2555.4	2011	5221.1
2000	1670.1	2006	2829.0	2012	5908.0

Firstly, the best translation step $\tau = 1149.9$ was obtained according to Equation (16).

Secondly, according to Equation (4), the parameters $[a, b]^{\mathrm{T}}$ were $[-0.1026, 850.0454]^{\mathrm{T}}$ from Table 2 and $[-0.1791, 205.7365]^{\mathrm{T}}$ from the raw data sequence translated by τ.

For the purpose of convenience of the analysis, the model GM(1,1) applied the translation transformation are defined as TT-GM(1,1). When the parameters $[a, b]^{\mathrm{T}}$ were put into Equation (8) and Equation (10), we could construct three grey models: GM(1,1) model, GGM(1,1) model and TT-GM(1,1) model, by which the 1995–2012 data could be estimated and the errors were given as the following Table 3.

According to the data from Table 3, the actual data compared with the predictive values obtained by using of three grey models: GM(1,1), TT-GM(1,1) and GGM(1,1) as Figure 1.

It can be seen from Table 3 that the average prediction model errors are much smaller by using of translation transformation, and we also find from Figure 1 that the improvement grey model TT-GM(1,1) is higher prediction accuracy comparing with other models.

Table 3 The forecasting results of the three grey models

Time	Actual value	GM(1,1)	ε	TT-GM(1,1)	ε	GGM(1,1)	ε
1995	1310.4	1310.4	0	1310.4	0	1310.4	0
1996	1572.1	1036.7	−535.42	1406.7	−165.44	1017.5	−554.58
1997	1617.2	1148.6	−468.56	1457	−160.19	1125.4	−491.82
1998	1590.3	1272.7	−317.62	1517.2	−73.062	1244.7	−345.63
1999	1577.4	1410.1	−167.27	1589.3	11.871	1376.6	−200.8
2000	1670.1	1562.4	−107.68	1675.4	5.3288	1522.5	−147.58
2001	1741.1	1731.2	−9.9491	1778.5	37.381	1683.9	−57.189
2002	1834.3	1918.1	83.808	1901.7	67.44	1862.4	28.106
2003	1943.3	2125.3	181.96	2049.2	105.87	2059.8	116.52
2004	2184.7	2354.8	170.07	2225.5	40.806	2278.2	93.462
2005	2555.4	2609.1	53.679	2436.4	−118.98	2519.6	−35.753
2006	2829	2890.8	61.849	2688.7	−140.31	2786.7	−42.27
2007	3223.9	3203	−20.852	2990.4	−233.47	3082.1	−141.78
2008	3660.7	3549	−111.74	3351.3	−309.36	3408.8	−251.87
2009	3993.5	3932.2	−61.263	3783	−210.49	3770.2	−223.34
2010	4381.8	4356.9	−24.898	4299.3	−82.474	4169.8	−212
2011	5221.1	4827.4	−393.67	4916.9	−304.21	4611.8	−609.3
2012	5908	5348.8	−559.23	5655.5	−252.46	5100.7	−807.35
$\bar{\varepsilon}$			184.9733		88.8413		242.1852
$\bar{\varepsilon}_r$			8.30%		3.54%		9.67%

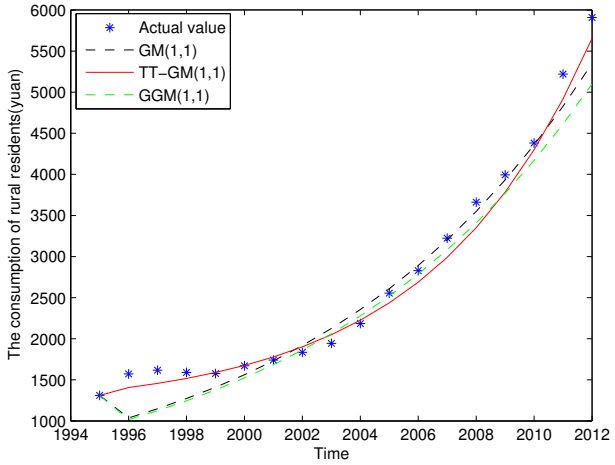


Figure 1 The actual data compared with the predictive values obtained by using of three grey models: GM(1,1), TT-GM(1,1) and GGM(1,1)

Finally, the development coefficient a in the TT-GM(1,1) model was -0.1791, and the average relative error $\bar{\varepsilon}_r$ which was 3.54% was lest among three models. According to the error test Table 1, we know that the TT-GM(1,1) model was highest accuracy among three models and almost achieved Class-I, so we could put the parameter $[a, b]^{\mathrm{T}} = [0.1791, 205.7365]^{\mathrm{T}}$ into Equation (8), then the consumption of rural residents in China from 2013 to 2016 which was forecasted by applying the TT-GM(1,1) model was [6539, 7595.8, 8859.7, 10372].

The actual consumption of rural residents in China 2013 is 6625.5 (yuan) from National Bureau of Statistics comparing with the predict value 6539, we know that the GM(1,1) model optimized by using of translation transformation can provide a very good predictive effect for the consumption of rural residents in theory. But there are many factors associated with the consumption of urban residents is unlikely to be sustained, so the TT-GM(1,1) model has a high accuracy when it is only applied to short-term prediction in fact.

5 Conclusions

In this paper, we improved grey modeling for a class growth sequence by the best translation transformation, and applied the improved grey model GM(1,1) to predict the consumption of rural residents in China. We can conclude that the best translation transformation is a simple method, and easy to implement, which can improve grey model accuracy very effectively. Meanwhile, we find the best translation method is particularly suitable for improving GM(1,1) model to forecast similar exponential increasing sequences such as: Economic development, population and the price growth, etc.

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