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一维含常数位势的非线性 Schrödinger 方程的概周期解

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摘要 在周期边界条件下, 考虑一维含常数位势 m 的非线性 Schrödinger 方程

$$\mathrm{i}u_t - \Delta u + mu + f(|u|^2)u = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{T},$$

其中 f 是原点某邻域内的一个实解析函数, 满足 $f(0) = 0$ 且 $f'(0) \neq 0$. 本文基于对频率漂移的分析、强 Diophantine 条件以及一种改进的无限维 KAM (Kolmogorov-Arnold-Moser) 理论, 证明对于任意给定的常数位势 m, 该方程存在一族 Whitney 光滑的小振幅、时间概周期解. 需要强调的是, 本文的解并非位于最大环面, 而是主要位于一个稀疏容许集上.

关键词 Schrödinger 方程 KAM 理论 概周期解 强 Diophantine 条件

MSC (2020) 主题分类 37K55, 35B10, 35J10, 35Q40, 35Q55

1 引言

Schrödinger 方程是 Hamilton 偏微分方程的典型模型, 可描述光学中的非线性波传播、量子力学系统及流体动力学中的一系列现象. 在非线性 Schrödinger 方程的动力系统分析中, 核心问题之一是“稳定” 解在非线性扰动下的持续性.

迄今为止, Hamilton 偏微分方程的 KAM (Kolmogorov-Arnold-Moser) 理论已取得诸多重要进展 (参见文献 [2, 25, 28–31, 34]). 其中值得关注的成果包括 KAM 方法的拓展 (参见文献 [8, 16, 20, 24, 36]), 以及 Craig, Wayne 和 Bourgain 逐步发展完善的 CWB (Craig-Wayne-Bourgain) 方法 (参见文献 [1, 4, 5, 11, 13, 15, 19, 37]). 这些研究成功证明了 Hamilton 扰动下拟周期解的存在性 (等价于有限维不变环面的持续性, 参见文献 [6, 21, 38, 39]). 相比之下, 关于概周期解的存在性研究—对应此类扰动下无穷维不变环面的持续性问题—仍不够深入, 相关研究可参见文献 [7, 23, 26, 27, 33, 35].

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在此背景下, 概周期解 (定义为拟周期函数一致极限的连续函数) 是一类关键解. 与依赖有限维不变环面的拟周期解不同, 概周期解天然与无限维不变环面相关联, 因而成为研究非线性 Schrödinger 方程无限维动力学行为的核心对象. 对于具有非多项式或缺乏精确可积性的非线性 Schrödinger 方程, 概周期解的存在性与正则性仍是未解难题, 二十年来的研究仅得到部分结论.

在构造非可积偏微分方程的概周期解方面, 早期进展很大程度上依赖外部参数 (如乘法位势或 Fourier 乘子) 来控制 "小除数"—这是 KAM 理论的本质难点, 源于共振频率间的相互作用. 偏微分方程领域的首个相关成果由 Bourgain [12] 提出, 他证实了带随机位势的波动方程存在概周期解. 数年之后, 他又在文献 [14] 中将该方法推广至带随机卷积位势的非线性 Schrödinger 方程, 证明了对绝大多数参数, 方程存在概周期解, 且其 Fourier 系数具有更加合理的衰减性. 此后, 一系列后续研究对该结果进行了拓展与完善, 包括文献 [9, 17] 的相关工作, 以及波动方程框架下的文献 [18].

Biasco 等 [10] 首次构造出适用于含外部参数的不可积一维非线性 Schrödinger 方程的弱概周期解. 这类弱解通过光滑测试函数的分布积分定义, 仅在时空上具备 Sobolev 正则性, 且多数弱解并非经典解 (即缺乏连续导数 u_t 和 u_{xx}). 该结果挑战了此前的主流范式, 即不可积偏微分方程的概周期解需要满足超快速 Fourier 衰减. 然而, 文献 [10] 仍依赖于外部参数 (Fourier 乘子 V) 调制频率以规避共振. 目前, 该领域的核心开放问题是, 能否通过调制有限个参数获得类似结果? 这一问题的解决将为更具物理关联性的偏微分方程的应用研究奠定基础. 尤为值得注意的是, 最具挑战性的场景出现在固定偏微分方程的情形, 此时上述参数需要直接从初始数据中 "提取".

正如 Bourgain [14] 指出的: "在无外部参数时, 非共振条件需通过振幅 - 频率调制及对作用变量的适当约束实现, 这一问题更为棘手. 实际上, 作用变量的快速衰减虽能提升过程收敛性, 却会减少频率调制空间, 加剧小除数问题."

Bernier 和 Grébert [3] 在近期研究中开始填补这一空白, 为无外部参数的非线性 Schrödinger 方程构造了非共振无限维 Kronecker 环面. 这类环面积累于 Kuksin 和 Pöschel [32] 提出的有限维 KAM 环面上, 且支撑非拟周期的概周期解. 文献 [3] 的关键创新在于正则化标准形—一种辛变换, 其利用非线性 Schrödinger 方程的色散特性 "平滑" 非线性项, 从而实现有限维环面向无限维的迭代扩展. 该方法通过内部提取参数, 无需依赖外部参数; 同时相继激发一组稀疏点集 ("切向位点") 简化小除数估计, 此模式对频率施加了强 Diophantine 条件.

Geng [22] 取得重要进展, 证明了周期或 Dirichlet 边界条件下一维非线性 Schrödinger 方程

$$\mathrm{i}u_t - u_{xx} + mu + f(|u|^2)u = 0$$

全维不变环面的存在性. 文献 [22] 通过结合四阶标准形约化与改进的 KAM 迭代方案, 构造出一族 Whitney 光滑的小振幅概周期解. 该证明的核心思想是, 证实紧致形式在 KAM 迭代中保持不变, 这一性质使得每一步 KAM 迭代仅需考虑本质的有限个小除数, 对测度估计具有重要意义. 然而, 该成果存在局限性: 定理对切向频率 (作为 ξ 的函数, 可视为独立参数) 施加了明确约束以控制小除数, 这一假设虽能有效证明存在性, 却限制了结果的一般性.

为填补上述空白, 本文以文献 [22] 的框架为基础, 旨在消除其对切向频率的明确约束条件. 实现这一目标的关键修改是, 充分利用切向点集 S 的稀疏性规避此类约束. 相应地, 本文构造的解不隶属于最大环面范畴, 仅主要分布于稀疏容许集内. 这使得这项工作迈出了第一步.

本文聚焦周期边界条件下的非线性 Schrödinger 方程

$$\mathrm{i}u_t - u_{xx} + mu + f(|u|^2)u = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{T}, \tag{1.1}$$

$$u(t,x)=u(t,x+2\pi),$$

其中 f 为原点邻域内定义的实解析函数, 满足 $f(0)=0$ 且 $f'(0)\neq 0$.

如前所述, 本文目标是从振幅 - 频率调制中提取参数. 核心挑战在于: 切向频率数量无限, 若同时处理所有调制频率, 将阻碍对保留频率的测度估计. 为此, 本文方法在每一步 KAM 迭代中仅处理有限个可控切向频率.

每一步迭代后, 需要通过进一步的作用 - 角变换从剩余法向坐标中提取参数. 然而, 传统方法将新的微小扰动限制在法向坐标的三阶, 而新调制频率的引入却需要四阶项, 形成难以调和的矛盾. 为解决这一问题, 本文修正 KAM 机制并强化小除数条件, 将新微小扰动提升至近似五阶, 从而为后续变换提供可行性.

标准形的构建涉及含 3 个或 4 个法向频率的小除数, 这将导致不同的小除数条件及更复杂的测度估计. 本文通过结合紧致形式扰动与切向点集 S 的固有稀疏性解决上述问题. 当扰动为五阶且标准形包含四阶平均项 (与第一步 KAM 迭代一致) 时, 作用 - 角变换的归纳步骤仍成立.

关键在于, 容许切向点集的选择可生成一组参数, 这类参数具有重要实用价值, 例如, 在受扰近可积系统分析中, 可有效规避共振, 并简化小除数 (KAM 迭代收敛的核心技术难题) 的估计. 下文陈述本文主要结果前, 先明确容许集的性质.

定义 1.1 设 $s:[0,+\infty)\to[0,+\infty)$ 是光滑的严格递增函数, 若无限点集 $S=\{i_1,\ldots,i_b,\ldots\}=s(\mathbb{N})\subset\mathbb{N}$ 满足以下条件:

(1) 存在 $j_*\geqslant 21$, 使得对所有 $j\geqslant j_*$, 有 $s(j)\geqslant \mathrm{e}^{(\log j)^{1+\iota}}$ (其中 $1<\iota\leqslant 2$);

(2) 对任意 $h\geqslant 1$ 及 $j,j'\geqslant j_*$, 有 $s(j+j')\geqslant s(j)+s(j')$ 且 $s(hj)\geqslant hs(j)$;

(3) 对任意 $j\geqslant j_*$, 有 $s(j^2)\geqslant s^2(j)$,

则称 S 是关于 s 的容许集 (admissible).

本文的主要结果如下:

定理 1.1 设 $S=\{i_1,\ldots,i_b,\ldots\}$ 是容许集, 则对任意 $0<\gamma\ll 1$, 存在 Cantor 集 $\tilde{\mathcal{O}}\subset\mathcal{O}$ 满足 $\mathrm{meas}(\mathcal{O}\setminus\tilde{\mathcal{O}})=O(\gamma)$, 使得对任意 $\xi=(\xi_1,\ldots,\xi_b,\ldots)\in\tilde{\mathcal{O}}$, 非线性 Schrödinger 方程 (1.1) 存在一族 Whitney 光滑的小振幅的时间概周期解, 形式为

$$u(t,x)=\sum_{j=1}^{\infty}\sqrt{\xi_j}\mathrm{e}^{\mathrm{i}(\omega_j t+nx)}+O(|\xi|^{\frac{3}{2}}),\quad \omega_j=i_j^2+m+O(|\xi|).$$

本文后续内容将围绕定理 1.1 的证明展开. 第 2 节为预备知识, 定义加权范数与紧致形式, 并分析其基本性质. 第 3 节针对方程 (1.1) 的 Hamilton 量, 推导得到四阶可积标准形, 进而完成参数提取. 第 4 节详细阐述 KAM 方法的单次迭代步骤. 第 5 节通过建立迭代引理、证明收敛性及完成测度估计, 最终完成定理 1.1 的证明.

2 预备知识

2.1 加权范数

$S_b=\{i_1,\ldots,i_b\}$ 是由 S 的前 b 个元素构成, 定义 $\mathbb{Z}_1^b=\mathbb{Z}\setminus\{i_1,\ldots,i_b\}$. 设 $z=(\ldots,z_n,\ldots)_{n\in\mathbb{Z}_1^b}$,

其复共轭为 $\bar{z}=(\ldots,\bar{z}_n,\ldots)_{n\in\mathbb{Z}_1^b}$. 定义加权范数

$$\|z\|_\rho=\sum_{n\in\mathbb{Z}_1^b}|z_n||n|^a\mathrm{e}^{|n|\rho},$$

其中 $\rho>0$, $a>1$. 将 $\mathbb{T}^b\times\{I=0\}\times\{z=0\}\times\{\bar{z}=0\}$ 的邻域定义为

$$D_\rho(r,s)=\{(\theta,I,z,\bar{z}):|\mathrm{Im}\theta|<r,|I|<s^2,\|z\|_\rho<s,\|\bar{z}\|_\rho<s\},$$

其中 $|\cdot|$ 表示复向量的上确界范数. 其他记号的定义类似. 设 $\mathcal{O}$ 表示 $\mathbb{R}_+^b$ 中一个具有正测度的参数集.

设 $\alpha\equiv(\ldots,\alpha_n,\ldots)_{n\in\mathbb{Z}_1^b}$, $\beta\equiv(\ldots,\beta_n,\ldots)_{n\in\mathbb{Z}_1^b}$, 其中 $\alpha_n,\beta_n\in\mathbb{N}$ 且仅有有限个非零分量. 记乘积 $z^\alpha\bar{z}^\beta$ 为 $\prod_n z_n^{\alpha_n}\bar{z}_n^{\beta_n}$. 对于函数

$$F(\theta,I,z,\bar{z})=\sum_{\alpha,\beta}F_{\alpha\beta}(\theta,I)z^\alpha\bar{z}^\beta,\tag{2.1}$$

其中 $F_{\alpha\beta}=\sum_{k\in\mathbb{Z}^b,l\in\mathbb{N}^b}F_{kl\alpha\beta}(\xi)I^l\mathrm{e}^{\mathrm{i}\langle k,\theta\rangle}$ 是关于参数 ξ 在 Whitney 意义下的 C_W^1 函数, 定义

$$\|F\|_{\mathcal{O}}=\sum_{\alpha,\beta,k,l}|F_{kl\alpha\beta}|_{\mathcal{O}}\ |I^l|\mathrm{e}^{|k||\mathrm{Im}\theta|}\ |z^\alpha||\bar{z}^\beta|,\tag{2.2}$$

其中

$$|F_{kl\alpha\beta}|_{\mathcal{O}}\equiv\sup_{\xi\in\mathcal{O}}\sum_{0\leqslant d\leqslant 1}|\partial_\xi^d F_{kl\alpha\beta}|$$

(所有关于参数 ξ 的导数均为 Whitney 意义下的导数). 定义 F 的加权范数[1)] 为

$$\|F\|_{D_\rho(r,s),\mathcal{O}}\equiv\sup_{D_\rho(r,s)}\|F\|_{\mathcal{O}}.\tag{2.3}$$

对于由函数 F 生成的向量场

$$X_F=(F_I,-F_\theta,\{\mathrm{i}F_{z_n}\}_{n\in\mathbb{Z}_1^b},\{-\mathrm{i}F_{\bar{z}_n}\}_{n\in\mathbb{Z}_1^b}),$$

定义其范数为

$$\|X_F\|_{D_\rho(r,s),\mathcal{O}}\equiv\|F_I\|_{D_\rho(r,s),\mathcal{O}}+\frac{1}{s^2}\|F_\theta\|_{D_\rho(r,s),\mathcal{O}}+\frac{1}{s}\sum_{n\in\mathbb{Z}_1^b}\|F_{z_n}\|_{\mathcal{O}}|n|^a\mathrm{e}^{|n|\rho}+\frac{1}{s}\sum_{n\in\mathbb{Z}_1^b}\|F_{\bar{z}_n}\|_{\mathcal{O}}|n|^a\mathrm{e}^{|n|\rho}.\tag{2.4}$$

2.2 紧致形式

定义在 $D_\rho(r,s)$ 上的实解析函数 $F=F(\theta,I,z,\bar{z})=\sum_{k,\alpha,\beta}F_{k\alpha\beta}(I)\mathrm{e}^{\mathrm{i}\langle k,\theta\rangle}q^\alpha\bar{q}^\beta$, 若对 $k=(k_1,k_2,\ldots,k_b)\in\mathbb{Z}^b$, $\alpha=(\ldots,\alpha_n,\ldots)$ 和 $\beta=(\ldots,\beta_n,\ldots)$ (其中 $\alpha_n,\beta_n\in\mathbb{N}$ 且仅有有限个分量非零), 当它们满足

$$\sum_{j=1}^b k_j i_j+\sum_n(\alpha_n-\beta_n)n\neq 0$$

时, 系数 $F_{k\alpha\beta}=0$, 则称 F 是具有紧致形式的函数.

1) (2.3), 对于标量函数定义了范数 $\|\cdot\|_{D_\rho(r,s),\mathcal{O}}$. 矢量函数 $G:D_\rho(r,s)\times\mathcal{O}\to\mathbb{C}^m$ $(m<\infty)$ 可类似定义其范数为 $\|G\|_{D_\rho(r,s),\mathcal{O}}=\sum_{i=1}^m\|G_i\|_{D_\rho(r,s),\mathcal{O}}$.

在本文中, Poisson 括号定义为

$$\{F,G\}=\left\langle\frac{\partial F}{\partial I},\frac{\partial G}{\partial\theta}\right\rangle-\left\langle\frac{\partial F}{\partial\theta},\frac{\partial G}{\partial I}\right\rangle+\mathrm{i}\sum_n\left(\frac{\partial F}{\partial q_n}\frac{\partial G}{\partial\bar{q}_n}-\frac{\partial F}{\partial\bar{q}_n}\frac{\partial G}{\partial q_n}\right).$$

证明本文主要结果的一个核心思路是, 紧致形式在 KAM 迭代过程中保持不变. 这一性质使得每一步只需要考虑有限个小除数, 对后续的测度估计至关重要.

引理 2.1 设 $F(\theta,I,q,\bar{q})$ 和 $G(\theta,I,q,\bar{q})$ 是 $D_\rho(r,s)$ 上的实解析函数. 若 F 和 G 均具有紧致形式, 则它们的 Poisson 括号 $\{F,G\}$ 也具有紧致形式.

本引理的证明可参见文献 [22].

3 标准形

在周期边界条件 $u(t,x)=u(t,x+2\pi)$ 下, 考虑一维非线性 Schrödinger 方程

$$\mathrm{i}u_t-\Delta u+mu+f(|u|^2)u=0,\quad t\in\mathbb{R},\quad x\in\mathbb{T},\tag{3.1}$$

其中 m 是常数位势, f 是定义在原点邻域内的实解析函数, 满足条件 $f(0)=0$ 且 $f'(0)\neq0$. 在周期边界条件下, 算子 $A=-\Delta+m$ 具有一组特征值 $\{\lambda_n\}$, 满足

$$\lambda_n=n^2+m,\quad n\in\mathbb{Z}.$$

此外, 对应的特征向量 $\phi_n(x)=\sqrt{\frac{1}{2\pi}}\mathrm{e}^{\mathrm{i}nx}$ 构成该算子在定义域上的一组基.

进一步地, 对非线性 Schrödinger 方程 (3.1) 作变换 $u=\mathrm{e}^{\mathrm{i}\sigma t}v$, 可得

$$\mathrm{i}v_t-\Delta v+(m-\sigma)v+f(|v|^2)v=0.$$

因此, 不失一般性, 后续讨论中可假设 $m=\frac{1}{5}$.

方程 (3.1) 可改写为 Hamilton 方程

$$u_t=\mathrm{i}\frac{\partial H}{\partial\bar{u}},\tag{3.2}$$

对应的 Hamilton 函数为

$$H=\langle Au,u\rangle+\int_0^{2\pi}g(|u|^2)dx,\tag{3.3}$$

其中 g 是 f 的原函数, $\langle\cdot,\cdot\rangle$ 表示 L^2 空间中的内积.

令

$$u(x,t)=\sum_{n\in\mathbb{Z}}q_n(t)\phi_n(x),$$

则 Hamilton 方程 (3.2) 等价于如下格点 Hamilton 方程:

$$\dot{q}_n=\mathrm{i}\left(\lambda_nq_n+\frac{\partial G}{\partial\bar{q}_n}\right),\quad G\equiv\int_0^{2\pi}g\left(\left|\sum_{n\in\mathbb{Z}}q_n\phi_n\right|^2\right)dx,\tag{3.4}$$

对应的 Hamilton 函数为

$$H=\sum_{n\in\mathbb{Z}}\lambda_n|q_n|^2+G.\tag{3.5}$$

如文献 [22,32] 所示, 方程 (3.4) 中 G 的正则性可由下述引理刻画.

引理 3.1 对任意 $\rho>0$, 梯度 $G_{\bar{q}}$ 是原点邻域内的实解析映射, 且满足

$$\|G_{\bar{q}}\|_\rho \leqslant c\|q\|_\rho^3. \tag{3.6}$$

证明 梯度 $G_{\bar{q}}=(\{\frac{\partial G}{\partial \bar{q}_n}\})$ 的解析性由其各分量的解析性及局部有界性可得. 设 $u=\sum_{n\in\mathbb{Z}} q_n\phi_n$, 易验证 $\frac{\partial G}{\partial \bar{q}_n}=(f(|u|^2)u)_n$. 因此,

$$\|G_{\bar{q}}\|_\rho=\|f(|u|^2)u\|_\rho\leqslant c\|u\|_\rho^3=c\|q\|_\rho^3.$$

证毕. □

已知 $G=\int_0^{2\pi} g(|\sum_{n\in\mathbb{Z}} q_n(t)\phi_n(x)|^2)dx=\sum_{\alpha,\beta} G_{\alpha\beta}q^\alpha\bar{q}^\beta$, 其中当 $\sum_n(\alpha_n-\beta_n)n\neq 0$ 时, $G_{\alpha\beta}=0$. 充分利用函数性质 $f(|u|^2)=c|u|^2+\text{h.o.t.}$ ("h.o.t." 表示 "高阶项"), 不失一般性, 可假设 $c=1$. 此时,

$$G=\frac{1}{2}\sum_{i,j,k,l} G_{ijkl}q_iq_j\bar{q}_k\bar{q}_l+\text{h.o.t.},$$

其中 $G_{ijkl}=\int_0^{2\pi}\phi_i\phi_j\phi_k\phi_l dx$.

对于初始切向点的容许集 $S_b=\{i_1,\dots,i_b\}\subset\mathbb{Z}$, 接下来需利用标准形技巧化简近可积系统.

命题 3.1 设 S_b 是初始选取的容许集, 则存在辛变换 Ψ 使得 Hamilton 函数 (3.5) 满足

$$H\circ\Psi=\langle\omega,I\rangle+\langle\Omega z,z\rangle+P, \tag{3.7}$$

其中

$$\begin{cases}\omega_j(\xi)=\varepsilon^{-8}\left(i_j^2+\dfrac{1}{5}\right)-\dfrac{1}{2\pi}\xi_j, & i_j\in S_b,\\ \Omega_n=\varepsilon^{-8}\left(n^2+\dfrac{1}{5}\right), & n\in\mathbb{Z}_1^b,\end{cases}$$

$$P=\acute{P}+\grave{P},$$

$$\acute{P}=-\sum_{i_j\in S_b}\frac{\varepsilon^2}{4\pi}I_j^2-\sum_{n\in\mathbb{Z}_1^b}\frac{\varepsilon^2}{4\pi}|z_n|^4,$$

$$\grave{P}=\varepsilon^6 O(\xi^3+\varepsilon\xi^{\frac{5}{2}}\|z\|_\rho+\varepsilon^2\xi^2\|z\|_\rho^2+\varepsilon^3\xi^{\frac{3}{2}}\|z\|_\rho^3+\varepsilon^4\xi\|z\|_\rho^4+\varepsilon^5\xi^{\frac{1}{2}}\|z\|_\rho^5+\varepsilon^6\|z\|_\rho^6).$$

证明 本证明依赖于变量的辛变换. 令

$$F_0=\sum_{\substack{i-j+k-l=0\\ i^2-j^2+k^2-l^2\neq 0\\ \sharp S_b\cap\{i,j,k,l\}\geqslant 2}}\frac{\mathrm{i}}{4\pi(\lambda_i-\lambda_j+\lambda_k-\lambda_l)}q_i\bar{q}_jq_k\bar{q}_l. \tag{3.8}$$

设 $\Phi_{F_0}^1$ 是对应 Hamilton 系统流的时间 1 映射. 变量替换 $\Phi_{F_0}^1$ 将 H 映射为

$$\begin{aligned}H\circ\Phi_{F_0}^1&=H+\{H,F_0\}+\int_0^1(1-t)\{\{H,F_0\},F_0\}\circ\Phi_{F_0}^t dt\\&=\sum_{i\in S_b}\lambda_i|q_i|^2+\sum_{n\in\mathbb{Z}_1^b}\lambda_n|z_n|^2+\sum_{i\in S_b}\frac{1}{4\pi}|q_i|^4\\&\quad+\sum_{i,j\in S_b,i\neq j}\frac{1}{\pi}|q_i|^2|q_j|^2+\sum_{i\in S_b,n\in\mathbb{Z}_1^b}\frac{1}{\pi}|q_i|^2|z_n|^2\end{aligned}$$

$$+O(\|z\|_\rho^4+\|q\|^6+|q|^5\|z\|_\rho+|q|^4\|z\|_\rho^2+|q|^3\|z\|_\rho^3+|q|^2\|z\|_\rho^4+|q|^1\|z\|_\rho^5+\|z\|_\rho^6).$$

对于 $n\in\mathbb{Z}_1^b$, 记 $z_n=q_n$, $\bar{z}_n=\bar{q}_n$. 注意到在后续迭代过程中, $\sum_n|q_n|^2$ 是守恒量, 且有

$$\frac{1}{2\pi}\left[\left(\sum_n|q_n|^2\right)^2-\frac{1}{2}\sum_n|q_n|^4\right]=\frac{1}{2\pi}C^2-\frac{1}{4\pi}\sum_n|q_n|^4.$$

因此, 忽略与后续分析无关的常数 C, 可得

$$\begin{aligned}H\circ\Phi_{F_0}^1=&\sum_{i\in S_b}\lambda_i|q_i|^2+\sum_{n\in\mathbb{Z}_1^b}\lambda_n|z_n|^2-\sum_{i\in S_b}\frac{1}{4\pi}|q_i|^4-\sum_{n\in\mathbb{Z}_1^b}\frac{1}{4\pi}|z_n|^4\\&+O(\|q\|^6+|q|^5\|z\|_\rho+|q|^4\|z\|_\rho^2+|q|^3\|z\|_\rho^3+|q|^2\|z\|_\rho^4+|q|^1\|z\|_\rho^5+\|z\|_\rho^6).\end{aligned}$$

在切空间中引入作用 - 角变量

$$q_{i_j}=\sqrt{I_j+\xi_j}\mathrm{e}^{\mathrm{i}\theta_j},\quad \bar{q}_{i_j}=\sqrt{I_j+\xi_j}\mathrm{e}^{-\mathrm{i}\theta_j},\quad i_j\in S_b,$$

则有

$$\begin{aligned}H\circ\Phi_{F_0}^1=&\sum_{i_j\in S_b}\lambda_j(I_j+\xi_j)+\sum_{n\in\mathbb{Z}_1^b}\lambda_n|z_n|^2-\sum_{i_j\in S_b}\frac{1}{4\pi}(I_j+\xi_j)^2-\sum_{n\in\mathbb{Z}_1^b}\frac{1}{4\pi}|z_n|^4\\&+O(\|q\|^6+|q|^5\|z\|_\rho+|q|^4\|z\|_\rho^2+|q|^3\|z\|_\rho^3+|q|^2\|z\|_\rho^4+|q|^1\|z\|_\rho^5+\|z\|_\rho^6)\\=&\sum_{i_j\in S_b}\lambda_jI_j+\sum_{n\in\mathbb{Z}_1^b}\lambda_n|z_n|^2-\sum_{i_j\in S_b}\frac{1}{2\pi}I_j\xi_j-\sum_{i_j\in S_b}\frac{1}{4\pi}I_j^2-\sum_{n\in\mathbb{Z}_1^b}\frac{1}{4\pi}|z_n|^4\\&+O(\xi^3+\xi^{\frac{5}{2}}\|z\|_\rho+\xi^2\|z\|_\rho^2+\xi^{\frac{3}{2}}\|z\|_\rho^3+\xi\|z\|_\rho^4+\xi^{\frac{1}{2}}\|z\|_\rho^5+\|z\|_\rho^6).\end{aligned}$$

进行尺度变换

$$\xi\to\varepsilon^8\xi,\quad I\to\varepsilon^{10}I,\quad\theta\to\theta,\quad z\to\varepsilon^5z,\quad\bar{z}\to\varepsilon^5\bar{z}.$$

最终得到的 Hamilton 函数为

$$\begin{aligned}H=&\varepsilon^{-18}H(\varepsilon^8\xi,\varepsilon^{10}I,\theta,\varepsilon^5z,\varepsilon^5\bar{z})\\=&\sum_{i_j\in S_b}\varepsilon^{-8}\lambda_jI_j+\sum_{n\in\mathbb{Z}_1^b}\varepsilon^{-8}\lambda_n|z_n|^2-\sum_{i_j\in S_b}\frac{1}{2\pi}I_j\xi_j-\sum_{i_j\in S_b}\frac{\varepsilon^2}{4\pi}I_j^2-\sum_{n\in\mathbb{Z}_1^b}\frac{\varepsilon^2}{4\pi}|z_n|^4\\&+O(\varepsilon^6\xi^3+\varepsilon^7\xi^{\frac{5}{2}}\|z\|_\rho+\varepsilon^8\xi^2\|z\|_\rho^2+\varepsilon^9\xi^{\frac{3}{2}}\|z\|_\rho^3+\varepsilon^{10}\xi\|z\|_\rho^4+\varepsilon^{11}\xi^{\frac{1}{2}}\|z\|_\rho^5+\varepsilon^{12}\|z\|_\rho^6)\\=&\langle\omega,I\rangle+\langle\Omega z,z\rangle+P,\end{aligned}$$

其中

$$\begin{cases}\omega_j(\xi)=\varepsilon^{-8}\left(i_j^2+\dfrac{1}{5}\right)-\dfrac{1}{2\pi}\xi_j, & i_j\in S_b,\\ \Omega_n=\varepsilon^{-8}\left(n^2+\dfrac{1}{5}\right), & n\in\mathbb{Z}_1^b,\end{cases}$$

$$P=\acute{P}+\grave{P},$$

$$\acute{P}=-\sum_{i_j\in S_b}\frac{\varepsilon^2}{4\pi}I_j^2-\sum_{n\in\mathbb{Z}_1^b}\frac{\varepsilon^2}{4\pi}|z_n|^4,$$

$$\grave{P} = \varepsilon^6 O(\xi^3 + \varepsilon\xi^{\frac{5}{2}}\|z\|_\rho + \varepsilon^2\xi^2\|z\|_\rho^2 + \varepsilon^3\xi^{\frac{3}{2}}\|z\|_\rho^3 + \varepsilon^4\xi\|z\|_\rho^4 + \varepsilon^5\xi^{\frac{1}{2}}\|z\|_\rho^5 + \varepsilon^6\|z\|_\rho^6).$$

证毕. □

注 3.1 在后续 KAM 迭代中, 会激发更多振子以增加切向频率的数量. 为得到更多参数, 需要利用振幅 - 频率调制来保证测度估计. 对于 $\acute{P}$ 中的项 $-\sum_{n\in\mathbb{Z}_1^b}\frac{\varepsilon^2}{4\pi}|z_n|^4$, 将把 $n = i_{b+1}$ 对应的变量 z_n 和 $\bar{z}_n$ 转化为作用 - 角变量, 且此过程中小除数条件需要始终成立.

设 $\mathcal{O} = [1,2]^b \subset \mathbb{R}_+^b$, $\gamma = \varepsilon^{\frac{3}{4}}$, $\tau = 25(b+1)$ 且 $\Omega = (\bar{\Omega}, \tilde{\Omega})$, 其中 $\bar{\Omega} = (\ldots, \Omega_n, \ldots)_{n=i_{b+1}}$, $\tilde{\Omega} = (\ldots, \Omega_n, \ldots)_{n\in\mathbb{Z}_1^{b+1}}$. 下一步将把 $\bar{\Omega}$ 转化为切向频率, 以增加切向频率的数量. 对所有满足 $\sum_{i_j\in S_b} k_j i_j + \sum_{n=i_{b+1}} d_n n + \sum_{n\in\mathbb{Z}_1^{b+1}} l_n n = 0$ 的整数向量 $(k,d,l) \in \mathbb{Z}^b \times \mathbb{Z} \times \mathbb{Z}^\infty$, 考虑集合

$$\mathcal{O}_0 = \Big\{\xi \in \mathcal{O} \Big| |\langle k, \omega(\xi)\rangle + \langle d, \bar{\Omega}\rangle + \langle l, \tilde{\Omega}\rangle| \geqslant \frac{\gamma}{\prod_{i_j\in S_b}(1+|k_j|^2 j^2(s))^\tau},$$
$$|k| + |d| + |l| \neq 0, |d| + |l| \leqslant 4, |l| \leqslant 3\Big\}.$$

其中 $j(s)$ 是定义 1.1 中光滑严格递增函数 $s(j)$ 的反函数.

为开始后续迭代, 取 $r, s, \rho > 0$, 考虑 $D(r,s)$ 上的 Hamilton 函数. 为保证小除数条件成立, 取 $\xi \in \mathcal{O}_0$. 令 $\omega^0 = \omega$, $\xi(0) = \xi_0 = \xi$, $I(0) = I = (I_n)_{n\in S_b}$, $\theta(0) = \theta = (\theta_n)_{n\in S_b}$, $\Omega_n^0 = \Omega_n$, $P_0 = P$, $r_0 = r$, $\gamma_0 = \gamma$, $\tau_0 = \tau$, 且

$$\begin{aligned} H_0 &= N_0 + P_0, \\ N_0 &= \langle \omega^0(\xi(0)), I(0)\rangle + \sum_{n\in\mathbb{Z}_1^b} \Omega_n^0(\xi(0)) z_n \bar{z}_n, \\ P_0 &= \acute{P}_0 + \grave{P}_0, \\ \acute{P}_0 &= -\sum_{i_j\in S_b} \frac{\varepsilon_0^2}{4\pi} I_j^2 - \sum_{n\in\mathbb{Z}_1^b} \frac{\varepsilon_0^2}{4\pi}|z_n|^4, \\ \grave{P}_0 &= O(\varepsilon_0^6). \end{aligned} \tag{3.9}$$

因此, H_0 在 $D_\rho(r_0, s)$ 上是实解析的, 且关于 $\xi(0) \in \mathcal{O}_0$ 在 Whitney 意义下光滑, 同时满足

$$\|X_{\acute{P}_0}\|_{D_\rho(r_0,s),\mathcal{O}_0}^{\bar{a}} \leqslant \varepsilon^2 = \varepsilon_0^2, \quad \|X_{\grave{P}_0}\|_{D_\rho(r_0,s),\mathcal{O}_0}^{\bar{a}} \leqslant \varepsilon^6 = \varepsilon_0^6,$$

其中 $\bar{a} = a - 1$.

对于接下来迭代中的切向点容许集 $S_{b+1} = \{i_1, \ldots, i_b, i_{b+1}\} \subset \mathbb{Z}$, 进一步整理标准形.

命题 3.2 设 S_{b+1} 是选取的容许集. 则存在辛变换 Φ_0, 使得 Hamilton 函数 (3.9) 满足

$$H_1 = H_0 \circ \Phi_0 = N_1 + P_1 = \langle \omega^1(\xi(0), \xi(1)), I(1)\rangle + \sum_{n\in\mathbb{Z}_1^{b+1}} \Omega_n^1(\xi(0), \xi(1)) z_n \bar{z}_n + \acute{P}_1 + \grave{P}_1,$$

其中

$$N_1 = \langle \omega^1(\xi(0), \xi(1)), I(1)\rangle + \sum_{n\in\mathbb{Z}_1^{b+1}} \Omega_n^1(\xi(0), \xi(1)) z_n \bar{z}_n,$$
$$\xi(0) = \xi, \quad \xi(1) = (\xi_n)_{n=i_{b+1}}, \quad I(1) = (I_n)_{n\in S_{b+1}},$$

$$\omega^1(\xi(0),\xi(1))=(\omega_0(\xi(0),\xi(1)),\omega_1(\xi(0),\xi(1))),$$
$$\omega_{0j}(\xi(0),\xi(1))=\varepsilon_0^{-8}\varepsilon_1^{-8}\left(|i_j|^2+\frac{1}{5}\right)-\frac{\varepsilon_1^{-8}}{2\pi}\xi_j+O(\varepsilon_0^6\varepsilon_1^{-8})f_0^1(\xi(0))+O(\varepsilon_0^6)f_1^1(\xi(1)),\quad i_j\in S_b,$$
$$\omega_1(\xi(0),\xi(1))=\varepsilon_0^{-8}\varepsilon_1^{-8}\left(|n|^2+\frac{1}{5}\right)-\frac{\varepsilon_0^2}{2\pi}\xi_n+O(\varepsilon_0^6\varepsilon_1^{-8})f_0^1(\xi(0))+O(\varepsilon_0^6)f_1^1(\xi(1)),\quad n=i_{b+1},$$
$$\Omega_n^1(\xi(0),\xi(1))=\varepsilon_0^{-8}\varepsilon_1^{-8}\left(|n|^2+\frac{1}{5}\right)+O(\varepsilon_0^6\varepsilon_1^{-8})f_0^1(\xi(0))+O(\varepsilon_0^6)f_1^1(\xi(1)),\quad n\in\mathbb{Z}_1^{b+1},$$
$$\|f_0^1(\xi(0))\|_{\mathcal{O}_1}=\sup_{\mathcal{O}_1}\left(|f_0^1(\xi(0))|+\left|\frac{\partial f_0^1}{\partial\xi(0)}\right|\right)\leqslant 1,$$
$$\|f_1^1(\xi(1))\|_{\mathcal{O}_1}=\sup_{\mathcal{O}_1}\left(|f_1^1(\xi(1))|+\left|\frac{\partial f_1^1}{\partial\xi(1)}\right|\right)\leqslant 1,$$
$$P_1=\acute{P}_1+\grave{P}_1,$$
$$\acute{P}_1=-\sum_{i_j\in S_{b+1}}\frac{\varepsilon_0^2\varepsilon_1^2}{4\pi}I_j^2-\sum_{n\in\mathbb{Z}_1^{b+1}}\frac{\varepsilon_0^2\varepsilon_1^2}{4\pi}|z_n|^4,$$
$$\grave{P}_1=O(\varepsilon_0^6\varepsilon_1^2).$$

证明 将扰动项展开为

$$P_0=\sum_{k,\alpha,\beta}P_{0k\alpha\beta}(I(0))\mathrm{e}^{\mathrm{i}\langle k,\theta(0)\rangle}z^\alpha\bar{z}^\beta,$$

需要强调: 当 $\sum_{i_j\in S_b}k_ji_j+\sum_{n\in\mathbb{Z}_1^b}(\alpha_n-\beta_n)n\neq 0$ 时, $P_{0k\alpha\beta}=0$. 令

$$F_1=\sum_{\substack{2|l|+|\bar{\alpha}+\bar{\beta}|\leqslant 4\\ 2|l|+|\tilde{\alpha}+\tilde{\beta}|\leqslant 3\\ |\bar{\alpha}+\bar{\beta}|+|\tilde{\alpha}+\tilde{\beta}|\leqslant 4}}\sum_{\substack{k\\ |k|+|\bar{\alpha}-\bar{\beta}|+|\tilde{\alpha}-\tilde{\beta}|\neq 0}}\frac{\mathrm{i}P_{0kl\bar{\alpha}\bar{\beta}\tilde{\alpha}\tilde{\beta}}}{\langle k,\omega^0(\xi(0))\rangle+\langle\bar{\alpha}-\bar{\beta},\bar{\Omega}^0\rangle+\langle\tilde{\alpha}-\tilde{\beta},\tilde{\Omega}^0\rangle}\mathrm{e}^{\mathrm{i}\langle k,\theta(0)\rangle}I(0)^l q^{\bar{\alpha}}\bar{q}^{\bar{\beta}}z^{\tilde{\alpha}}\bar{z}^{\tilde{\beta}}.\quad(3.10)$$

F_1 在时间 1 处的 Hamilton 流映射定义了一个辛映射 $\Phi_{F_1}^1$. 在该辛映射下,

$$\begin{aligned}H_0\circ\Phi_{F_1}^1&=H_0+\{H_0,F_1\}+\int_0^1(1-t)\{\{H_0,F_1\},F_1\}\circ\Phi_{F_1}^t dt\\&=\sum_{i_j\in S_b}\varepsilon_0^{-8}\lambda_jI_j+\sum_{n=i_{b+1}}\varepsilon_0^{-8}\lambda_n|z_n|^2+\sum_{n\in\mathbb{Z}_1^{b+1}}\varepsilon_0^{-8}\lambda_n|z_n|^2-\sum_{i_j\in S_b}\frac{1}{2\pi}I_j\xi_j\\&\quad-\sum_{i_j\in S_b}\frac{\varepsilon_0^2}{4\pi}I_j^2-\sum_{n=i_{b+1}}\frac{\varepsilon_0^2}{4\pi}|z_n|^4-\sum_{n\in\mathbb{Z}_1^{b+1}}\frac{\varepsilon_0^2}{4\pi}|z_n|^4\\&\quad+O(\varepsilon_0^6\xi_0^3+\varepsilon_0^{10}\xi_0\|z\|_\rho^4+\varepsilon_0^{11}\xi_0^{\frac{1}{2}}\|z\|_\rho^5+\varepsilon_0^{12}\|z\|_\rho^6).\end{aligned}$$

激发满足 $n=i_{b+1}$ 的变量 z_n 和 $\bar{z}_n$, 并将其转化为作用 - 角变量

$$z_n=\sqrt{I_n+\xi_n}\mathrm{e}^{\mathrm{i}\theta_n},\quad \bar{z}_n=\sqrt{I_n+\xi_n}\mathrm{e}^{-\mathrm{i}\theta_n},\quad n=i_{b+1},$$

此时有

$$H_0\circ\Phi_{F_1}^1=\sum_{i_j\in S_{b+1}}\varepsilon_0^{-8}\lambda_jI_j+\sum_{n\in\mathbb{Z}_1^{b+1}}\varepsilon_0^{-8}\lambda_n|z_n|^2-\sum_{i_j\in S_b}\frac{1}{2\pi}I_j\xi_j-\sum_{n=i_{b+1}}\frac{\varepsilon_0^2}{2\pi}I_n\xi_n$$

$$
\begin{aligned}
&-\sum_{i_j\in S_b}\frac{\varepsilon_0^2}{4\pi}I_j^2-\sum_{n=i_{b+1}}\frac{\varepsilon_0^2}{4\pi}I_n^2-\sum_{n\in\mathbb{Z}_1^{b+1}}\frac{\varepsilon_0^2}{4\pi}|z_n|^4\\
&+O(\varepsilon_0^6\xi_0^3+\varepsilon_0^{10}\xi_0\|z\|_\rho^4+\varepsilon_0^{11}\xi_0^{\frac{1}{2}}\|z\|_\rho^5+\varepsilon_0^{12}\|z\|_\rho^6).
\end{aligned}
$$

进行尺度变换

$$
\xi_n\to\varepsilon_1^8\xi_n,\quad I\to\varepsilon_1^{10}I,\quad \theta\to\theta,\quad z\to\varepsilon_1^5 z,\quad \bar z\to\varepsilon_1^5\bar z.
$$

最终得到的 Hamilton 函数为

$$
\begin{aligned}
H_1&=\varepsilon_1^{-18}H_1(\varepsilon_1^8\xi_n,\varepsilon_1^{10}I,\theta,\varepsilon_1^5 z,\varepsilon_1^5\bar z)\\
&=\sum_{i_j\in S_{b+1}}\varepsilon_0^{-8}\varepsilon_1^{-8}\lambda_j I_j+\sum_{n\in\mathbb{Z}_1^{b+1}}\varepsilon_0^{-8}\varepsilon_1^{-8}\lambda_n|z_n|^2-\sum_{i_j\in S_b}\frac{\varepsilon_1^{-8}}{2\pi}I_j\xi_j-\sum_{n=i_{b+1}}\frac{\varepsilon_0^2}{2\pi}I_n\xi_n\\
&\quad-\sum_{i_j\in S_{b+1}}\frac{\varepsilon_0^2\varepsilon_1^2}{4\pi}I_j^2-\sum_{n\in\mathbb{Z}_1^{b+1}}\frac{\varepsilon_0^2\varepsilon_1^2}{4\pi}|z_n|^4+O(\varepsilon_0^6\varepsilon_1^2)(|I|^2+\|z\|_\rho^4+|I|\|z\|_\rho^2)\\
&=N_1+P_1=\langle\omega^1(\xi(0),\xi(1)),I(1)\rangle+\sum_{n\in\mathbb{Z}_1^{b+1}}\Omega_n^1(\xi(0),\xi(1))z_n\bar z_n+\acute{P}_1+\grave{P}_1,
\end{aligned}
$$

其中

$$
\begin{aligned}
&N_1=\langle\omega^1(\xi(0),\xi(1)),I(1)\rangle+\sum_{n\in\mathbb{Z}_1^{b+1}}\Omega_n^1(\xi(0),\xi(1))z_n\bar z_n,\\
&\xi(0)=\xi,\quad \xi(1)=(\xi_n)_{n=i_{b+1}},\quad I(1)=(I_n)_{n\in S_{b+1}},\\
&\omega^1(\xi(0),\xi(1))=(\omega_0(\xi(0),\xi(1)),\omega_1(\xi(0),\xi(1))),\\
&\omega_{0j}(\xi(0),\xi(1))=\varepsilon_0^{-8}\varepsilon_1^{-8}\left(|i_j|^2+\frac{1}{5}\right)-\frac{\varepsilon_1^{-8}}{2\pi}\xi_j+O(\varepsilon_0^6\varepsilon_1^{-8})f_0^1(\xi(0))+O(\varepsilon_0^6)f_1^1(\xi(1)),\quad i_j\in S_b,\\
&\omega_1(\xi(0),\xi(1))=\varepsilon_0^{-8}\varepsilon_1^{-8}\left(|n|^2+\frac{1}{5}\right)-\frac{\varepsilon_0^2}{2\pi}\xi_n+O(\varepsilon_0^6\varepsilon_1^{-8})f_0^1(\xi(0))+O(\varepsilon_0^6)f_1^1(\xi(1)),\quad n=i_{b+1},\\
&\Omega_n^1(\xi(0),\xi(1))=\varepsilon_0^{-8}\varepsilon_1^{-8}\left(|n|^2+\frac{1}{5}\right)+O(\varepsilon_0^6\varepsilon_1^{-8})f_0^1(\xi(0))+O(\varepsilon_0^6)f_1^1(\xi(1)),\quad n\in\mathbb{Z}_1^{b+1},\\
&\|f_0^1(\xi(0))\|_{\mathcal{O}_1}=\sup_{\mathcal{O}_1}\left(|f_0^1(\xi(0))|+\left|\frac{\partial f_0^1}{\partial\xi(0)}\right|\right)\leqslant 1,\\
&\|f_1^1(\xi(1))\|_{\mathcal{O}_1}=\sup_{\mathcal{O}_1}\left(|f_1^1(\xi(1))|+\left|\frac{\partial f_1^1}{\partial\xi(1)}\right|\right)\leqslant 1,\\
&P_1=\acute{P}_1+\grave{P}_1,\\
&\acute{P}_1=-\sum_{i_j\in S_{b+1}}\frac{\varepsilon_0^2\varepsilon_1^2}{4\pi}I_j^2-\sum_{n\in\mathbb{Z}_1^{b+1}}\frac{\varepsilon_0^2\varepsilon_1^2}{4\pi}|z_n|^4,\\
&\grave{P}_1=O(\varepsilon_0^6\varepsilon_1^2).
\end{aligned}
$$

证毕. □

H_1 在 $(\theta(1),I(1),z,\bar z)\in D_1=D_{\rho_1}(r_1,s_1)$ 上是实解析的, 且关于 $(\xi(0),\xi(1))\in\mathcal{O}_1\subset[1,2]^{b+1}$ 在

Whitney 意义下光滑, 同时满足 $\sum_{i_j\in S_{b+1}} k_j i_j + \sum_{n=i_{b+2}} d_n n + \sum_{n\in\mathbb{Z}_1^{b+2}} l_n n = 0$. 参数集为

$$\mathcal{O}_1 = \left\{ (\xi(0),\xi(1)) \in \mathcal{O}_0 \times [1,2] \,\middle|\, \begin{array}{l} |\langle k,\omega^1(\xi(0),\xi(1))\rangle + \langle d,\bar{\Omega}^1\rangle + \langle l,\tilde{\Omega}^1\rangle| \geqslant \dfrac{\gamma_1}{\prod_{i_j\in S_{b+1}}(1+|k_j|^2 j^2(s))^{\tau_1}}, \\ |k|+|d|+|l|\neq 0, |d|+|l|\leqslant 4, |l|\leqslant 3 \end{array} \right\}.$$

此外, 有

$$\|X_{\acute{P}_1}\|^{\bar{a}}_{D_{\rho_1}(r_1,s_1),\mathcal{O}_1} \leqslant \varepsilon_0^2\varepsilon_1^2, \quad \|X_{\grave{P}_1}\|^{\bar{a}}_{D_{\rho_1}(r_1,s_1),\mathcal{O}_1} \leqslant \varepsilon_0^6\varepsilon_1^2.$$

4 KAM 步骤

KAM 迭代算法的构建: 在每一个迭代步骤中, 通过坐标变换消除包含角变量的不可积分量 (这些变换必须保持紧致形式的结构特性). 这确保了当扰动参数趋近于零时, 新的扰动项成为前一步扰动项的高阶无穷小.

收敛域的构建: 在每一次迭代中, 会从作用变量的定义域中排除可能导致迭代发散的取值集合. 最终, 这保证了迭代在一个具有正测度的集合上收敛.

简而言之, KAM 迭代过程的核心如下: 在排除参数集中的一个小子集并控制范数权重收缩的条件下, 使每一步的扰动项成为前一步扰动项的高阶无穷小. 最终证明, 无限步的 KAM 迭代是收敛的, 且余下的参数集合具有近乎满测度.

第 ν 步 KAM 迭代, Hamilton 函数

$$H_\nu = N_\nu + P_\nu,$$

其中

$$\begin{aligned}
&N_\nu = \langle \omega^\nu(\xi(0),\ldots,\xi(\nu)), I(\nu)\rangle + \sum_{n\in\mathbb{Z}_1^{b+\nu}} \Omega_n^\nu(\xi(0),\ldots,\xi(\nu)) z_n \bar{z}_n,\\
&\xi(0)=\xi, \quad \xi(1)=(\xi_n)_{n=i_{b+1}}, \quad \ldots, \quad \xi(\nu)=(\xi_n)_{n=i_{b+\nu}},\\
&I(\nu)=(I_n)_{n\in S_{b+\nu}}, \quad \theta(\nu)=(\theta_n)_{n\in S_{b+\nu}},\\
&\omega^\nu(\xi(0),\ldots,\xi(\nu)) = (\omega_0(\xi(0),\ldots,\xi(\nu)),\ldots,\omega_\nu(\xi(0),\ldots,\xi(\nu))),\\
&\omega_{0j}(\xi(0),\ldots,\xi(\nu)) = \varepsilon_0^{-8}\cdots\varepsilon_\nu^{-8}\left(|i_j|^2+\frac{1}{5}\right) - \frac{\varepsilon_1^{-8}\cdots\varepsilon_\nu^{-8}}{2\pi}\xi_j\\
&\qquad\qquad + O(\varepsilon_0^6\varepsilon_1^{-8}\cdots\varepsilon_\nu^{-8}) f_0^\nu(\xi(0)) + O(\varepsilon_0^6\varepsilon_2^{-8}\cdots\varepsilon_\nu^{-8}) f_1^\nu(\xi(1))\\
&\qquad\qquad + \cdots + O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_{i-1}^2\varepsilon_{i+1}^{-8}\cdots\varepsilon_\nu^{-8}) f_i^\nu(\xi(i))\\
&\qquad\qquad + \cdots + O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_{\nu-1}^2) f_\nu^\nu(\xi(\nu)), \quad i_j\in S_b,\\
&\omega_n(\xi(0),\ldots,\xi(\nu)) = \varepsilon_0^{-8}\cdots\varepsilon_\nu^{-8}\left(|i_{b+n}|^2+\frac{1}{5}\right) - \frac{\varepsilon_0^2\varepsilon_1^2\cdots\varepsilon_{n-1}^2\varepsilon_{n+1}^{-8}\cdots\varepsilon_\nu^{-8}}{2\pi}\xi_n\\
&\qquad\qquad + O(\varepsilon_0^6\varepsilon_1^{-8}\cdots\varepsilon_\nu^{-8}) f_0^\nu(\xi(0)) + O(\varepsilon_0^6\varepsilon_2^{-8}\cdots\varepsilon_\nu^{-8}) f_1^\nu(\xi(1))\\
&\qquad\qquad + \cdots + O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_{i-1}^2\varepsilon_{i+1}^{-8}\cdots\varepsilon_\nu^{-8}) f_i^\nu(\xi(i))\\
&\qquad\qquad + \cdots + O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_{\nu-1}^2) f_\nu^\nu(\xi(\nu)), \quad n=1,2,\ldots,\nu,
\end{aligned}$$

$$
\begin{aligned}
\Omega_n^\nu(\xi(0),\dots,\xi(\nu)) &= \varepsilon_0^{-8}\cdots\varepsilon_\nu^{-8}\left(|n|^2+\frac{1}{5}\right)\\
&\quad + O(\varepsilon_0^6\varepsilon_1^{-8}\cdots\varepsilon_\nu^{-8})f_0^\nu(\xi(0)) + O(\varepsilon_0^6\varepsilon_2^{-8}\cdots\varepsilon_\nu^{-8})f_1^\nu(\xi(1))\\
&\quad + \cdots + O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_{i-1}^2\varepsilon_{i+1}^{-8}\cdots\varepsilon_\nu^{-8})f_i^\nu(\xi(i))\\
&\quad + \cdots + O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_{\nu-1}^2)f_\nu^\nu(\xi(\nu)),\quad n\in\mathbb{Z}_1^{b+\nu},\\
\|f_i^\nu(\xi(i))\|_{\mathcal{O}_\nu} &= \sup_{\mathcal{O}_\nu}\left(|f_i^\nu(\xi(i))| + \left|\frac{\partial f_i^\nu}{\partial\xi(i)}\right|\right)\leqslant 1,\quad i=0,1,\dots,\nu,\\
P_\nu &= \acute{P}_\nu + \grave{P}_\nu,\\
\acute{P}_\nu &= -\sum_{i_j\in S_{b+\nu}}\frac{\varepsilon_0^2\varepsilon_1^2\cdots\varepsilon_\nu^2}{4\pi}I_j^2 - \sum_{n\in\mathbb{Z}_1^{b+\nu}}\frac{\varepsilon_0^2\varepsilon_1^2\cdots\varepsilon_\nu^2}{4\pi}|z_n|^4,\\
\grave{P}_\nu &= O(\varepsilon_0^6\varepsilon_1^2\varepsilon_2^2\cdots\varepsilon_\nu^2).
\end{aligned}
$$

H_ν 在 $(\theta(\nu),I(\nu),z,\bar z)\in D_\nu = D_{\rho_\nu}(r_\nu,s_\nu)$ 上是实解析的, 且关于 $(\xi(0),\xi(1),\dots,\xi(\nu))\in\mathcal{O}_\nu\subset[1,2]^{b+\nu}$ 在 Whitney 意义下光滑, 同时满足 $\sum_{i_j\in S_{b+\nu}}k_ji_j+\sum_{n=i_{b+\nu+1}}d_nn+\sum_{n\in\mathbb{Z}_1^{b+\nu+1}}l_nn=0$. 参数集 $\mathcal{O}_\nu$ 定义为

$$
\left\{(\xi(0),\dots,\xi(\nu))\in\mathcal{O}_{\nu-1}\times[1,2]\;\middle|\;\begin{array}{l}|\langle k,\omega^\nu(\xi(0),\dots,\xi(\nu))\rangle+\langle d,\bar\Omega^\nu\rangle+\langle l,\tilde\Omega^\nu\rangle|\geqslant\dfrac{\gamma_\nu}{\prod_{i_j\in S_{b+\nu}}(1+|k_j|^2j^2(s))^{\tau_\nu}},\\ |k|+|d|+|l|\neq0,|d|+|l|\leqslant4,|l|\leqslant3\end{array}\right\}.
$$

此外, 成立

$$
\|X_{\acute P_\nu}\|^{\bar a}_{D_{\rho_\nu}(r_\nu,s_\nu),\mathcal{O}_\nu}\leqslant\prod_{i=0}^{\nu}\varepsilon_i^2,\quad \|X_{\grave P_\nu}\|^{\bar a}_{D_{\rho_\nu}(r_\nu,s_\nu),\mathcal{O}_\nu}\leqslant\varepsilon_0^4\prod_{i=0}^{\nu}\varepsilon_i^2.
$$

在下一小节中, 通过求解同调方程来构建映射

$$
\Phi_v : D_{\rho_\nu}(r_{\nu+1},s_{\nu+1})\times\mathcal{O}_\nu\to D_{\rho_\nu}(r_\nu,s_\nu)\times\mathcal{O}_\nu,
$$

使得

$$
H_{\nu+1} = H_\nu\circ\Phi_\nu = N_{\nu+1}+P_{\nu+1},\tag{4.1}
$$

这也满足 $D_{\rho_{\nu+1}}(r_{\nu+1},s_{\nu+1})\times\mathcal{O}_{\nu+1}$ 上的迭代假设. 此外,

$$
\|X_{\acute P_{\nu+1}}\|^{\bar a}_{D_{\rho_{\nu+1}}(r_{\nu+1},s_{\nu+1}),\mathcal{O}_{\nu+1}}\leqslant\prod_{i=0}^{\nu+1}\varepsilon_i^2,\quad \|X_{\grave P_{\nu+1}}\|^{\bar a}_{D_{\rho_{\nu+1}}(r_{\nu+1},s_{\nu+1}),\mathcal{O}_{\nu+1}}\leqslant\varepsilon_0^4\prod_{i=0}^{\nu+1}\varepsilon_i^2.
$$

4.1 同调方程

展开 $\grave P_\nu$, 可得

$$
\grave P_\nu = \sum_{k\in\mathbb{Z}^{\nu+b},l\in\mathbb{N}^{\nu+b},\alpha\in\mathbb{N}^{\mathbb{Z}_1^{\nu+b}},\beta\in\mathbb{N}^{\mathbb{Z}_1^{\nu+b}}}\grave P_{\nu kl\alpha\beta}(I(\nu))\mathrm{e}^{\mathrm{i}\langle k,\theta(\nu)\rangle}z^\alpha\bar z^\beta.
$$

令 $q=(\dots,z_n,\dots)_{n=i_{\nu+b+1}}$ 和 $\bar q=(\dots,\bar z_n,\dots)_{n=i_{\nu+b+1}}$, 并截断 $\grave P_\nu$ 为

$$
R_\nu(I(\nu),\theta(\nu),q,\bar q,z,\bar z) = \sum_{\substack{2|l|+|\bar\alpha+\bar\beta|\leqslant4\\ 2|l|+|\bar\alpha+\bar\beta|\leqslant3\\ |\bar\alpha+\bar\beta|+|\bar\alpha+\bar\beta|\leqslant4}}\sum_k\grave P_{\nu kl\bar\alpha\bar\beta\tilde\alpha\tilde\beta}\mathrm{e}^{\mathrm{i}\langle k,\theta(\nu)\rangle}I(\nu)^lq^{\bar\alpha}\bar q^{\bar\beta}z^{\tilde\alpha}\bar z^{\tilde\beta}.
$$

定义

$$[R_\nu]=\sum_{\substack{|l|+|\bar\alpha|\leqslant 2\\ |l|+|\tilde\alpha|\leqslant 1\\ |\bar\alpha|+|\tilde\alpha|\leqslant 2}}\grave{P}_{\nu 0l\bar\alpha\bar\beta\tilde\alpha\tilde\beta}I(\nu)^l q^{\bar\alpha}\bar q^{\bar\beta}z^{\tilde\alpha}\bar z^{\tilde\beta},\tag{4.2}$$

在激发下一步新的切向频率前, 需要在 $D_{\rho_{\nu+1}}(r_{\nu+1},s_{\nu+1})\times\mathcal{O}_{\nu+1}$ 上构造函数 F_ν, 使得其 Hamilton 向量场 X_{F_ν} 的时间 1 流映射 $\Phi^1_{F_\nu}$ 能够定义一个从 $D_{\nu+1}$ 到 D_ν 的映射. 这类 F_ν 可选取紧致形式的构造

$$F_\nu(I(\nu),\theta(\nu),q,\bar q,z,\bar z)=\sum_{\substack{2|l|+|\bar\alpha+\bar\beta|\leqslant 4\\ 2|l|+|\tilde\alpha+\tilde\beta|\leqslant 3\\ |\bar\alpha+\bar\beta|+|\tilde\alpha+\tilde\beta|\leqslant 4}}\ \sum_{\substack{k\\ |k|+|\bar\alpha-\bar\beta|+|\tilde\alpha-\tilde\beta|\neq 0}}F_{\nu kl\bar\alpha\bar\beta\tilde\alpha\tilde\beta}\mathrm{e}^{\mathrm{i}\langle k,\theta(\nu)\rangle}I(\nu)^l q^{\bar\alpha}\bar q^{\bar\beta}z^{\tilde\alpha}\bar z^{\tilde\beta},$$

当

$$\sum_{i_j\in S_{b+\nu}}k_ji_j+\sum_{n=i_{b+\nu+1}}(\bar\alpha_n-\bar\beta_n)n+\sum_{n\in\mathbb{Z}_1^{b+\nu+1}}(\tilde\alpha_n-\tilde\beta_n)n\neq 0$$

时, 有 $F_{k\alpha\beta}=0$, 并且满足同调方程

$$\{N_\nu,F_\nu\}+R_\nu=[R_\nu].\tag{4.3}$$

通过比较系数, 可知同调方程 (4.3) 等价于

$$(\langle k,\omega^\nu\rangle+\langle\bar\alpha-\bar\beta,\bar\Omega^\nu\rangle+\langle\tilde\alpha-\tilde\beta,\tilde\Omega^\nu\rangle)F_{\nu kl\bar\alpha\bar\beta\tilde\alpha\tilde\beta}=\mathrm{i}\grave{P}_{\nu kl\bar\alpha\bar\beta\tilde\alpha\tilde\beta}.\tag{4.4}$$

4.2 激发切向频率

将满足 $n=i_{b+\nu+1}$ 的变量 z_n 和 $\bar z_n$ 激发, 转化为作用 - 角变量

$$z_n=\sqrt{I_n+\xi_n}\mathrm{e}^{\mathrm{i}\theta_n},\quad \bar z_n=\sqrt{I_n+\xi_n}\mathrm{e}^{-\mathrm{i}\theta_n},\quad n=i_{b+\nu+1}.$$

进行尺度变换

$$\xi_n\to\varepsilon_1^8\xi_n,\quad I\to\varepsilon_1^{10}I,\quad \theta\to\theta,\quad z\to\varepsilon_1^5 z,\quad \bar z\to\varepsilon_1^5\bar z,$$

得到 Hamilton 函数

$$H_{\nu+1}=N_{\nu+1}+P_{\nu+1},$$

其中

$$\begin{aligned}
&N_{\nu+1}=\langle\omega^\nu(\xi(0),\dots,\xi(\nu+1)),I(\nu+1)\rangle+\sum_{n\in\mathbb{Z}_1^{b+\nu+1}}\Omega_n^{\nu+1}(\xi(0),\dots,\xi(\nu+1))z_n\bar z_n,\\
&\xi(0)=\xi,\quad \xi(1)=(\xi_n)_{n=i_{b+1}},\quad \dots,\quad \xi(\nu+1)=(\xi_n)_{n=i_{b+\nu+1}},\\
&I(\nu+1)=(I_n)_{n\in S_{b+\nu+1}},\quad \theta(\nu+1)=(\theta_n)_{n\in S_{b+\nu+1}},\\
&\omega^{\nu+1}(\xi(0),\dots,\xi(\nu+1))=(\omega_0(\xi(0),\dots,\xi(\nu+1)),\dots,\omega_{\nu+1}(\xi(0),\dots,\xi(\nu+1))),\\
&\omega_{0j}(\xi(0),\dots,\xi(\nu+1))=\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}\left(|i_j|^2+\frac{1}{5}\right)-\frac{\varepsilon_1^{-8}\cdots\varepsilon_{\nu+1}^{-8}}{2\pi}\xi_j\\
&\qquad\qquad+O(\varepsilon_0^6\varepsilon_1^{-8}\cdots\varepsilon_{\nu+1}^{-8})f_0^{\nu+1}(\xi(0))+O(\varepsilon_0^6\varepsilon_2^{-8}\cdots\varepsilon_{\nu+1}^{-8})f_1^{\nu+1}(\xi(1))\\
&\qquad\qquad+\cdots+O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_{i-1}^2\varepsilon_{i+1}^{-8}\cdots\varepsilon_{\nu+1}^{-8})f_i^{\nu+1}(\xi(i))\\
&\qquad\qquad+\cdots+O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_\nu^2)f_{\nu+1}^{\nu+1}(\xi(\nu+1)),\quad i_j\in S_b,
\end{aligned}$$

$$\begin{aligned}\omega_n(\xi(0),\dots,\xi(\nu+1)) &= \varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}\left(|i_{b+n}|^2+\frac{1}{5}\right)-\frac{\varepsilon_0^2\varepsilon_1^2\cdots\varepsilon_{n-1}^2\varepsilon_{n+1}^{-8}\cdots\varepsilon_{\nu+1}^{-8}}{2\pi}\xi_n\\ &\quad+O(\varepsilon_0^6\varepsilon_1^{-8}\cdots\varepsilon_{\nu+1}^{-8})f_0^{\nu+1}(\xi(0))+O(\varepsilon_0^6\varepsilon_2^{-8}\cdots\varepsilon_{\nu+1}^{-8})f_1^{\nu+1}(\xi(1))\\ &\quad+\cdots+O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_{i-1}^2\varepsilon_{i+1}^{-8}\cdots\varepsilon_{\nu+1}^{-8})f_i^{\nu+1}(\xi(i))\\ &\quad+\cdots+O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_\nu^2)f_{\nu+1}^{\nu+1}(\xi(\nu+1)),\quad n=1,\dots,\nu+1,\\ \Omega_n^{\nu+1}(\xi(0),\dots,\xi(\nu+1)) &= \varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}\left(|n|^2+\frac{1}{5}\right)\\ &\quad+O(\varepsilon_0^6\varepsilon_1^{-8}\cdots\varepsilon_{\nu+1}^{-8})f_0^{\nu+1}(\xi(0))+O(\varepsilon_0^6\varepsilon_2^{-8}\cdots\varepsilon_{\nu+1}^{-8})f_1^{\nu+1}(\xi(1))\\ &\quad+\cdots+O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_{i-1}^2\varepsilon_{i+1}^{-8}\cdots\varepsilon_{\nu+1}^{-8})f_i^{\nu+1}(\xi(i))\\ &\quad+\cdots+O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_\nu^2)f_{\nu+1}^{\nu+1}(\xi(\nu+1)),\quad n\in\mathbb{Z}_1^{b+\nu+1},\\ \|f_i^{\nu+1}(\xi(i))\|_{\mathcal{O}_{\nu+1}} &= \sup_{\mathcal{O}_{\nu+1}}\left(|f_i^{\nu+1}(\xi(i))|+\left|\frac{\partial f_i^{\nu+1}}{\partial\xi(i)}\right|\right)\leqslant 1,\quad i=0,1,\dots,\nu+1,\\ P_{\nu+1} &= \acute{P}_{\nu+1}+\grave{P}_{\nu+1},\\ \acute{P}_{\nu+1} &= -\sum_{i_j\in S_{b+\nu+1}}\frac{\varepsilon_0^2\varepsilon_1^2\cdots\varepsilon_{\nu+1}^2}{4\pi}I_j^2-\sum_{n\in\mathbb{Z}_1^{b+\nu+1}}\frac{\varepsilon_0^2\varepsilon_1^2\cdots\varepsilon_{\nu+1}^2}{4\pi}|z_n|^4,\\ \grave{P}_{\nu+1} &= O(\varepsilon_0^6\varepsilon_1^2\varepsilon_2^2\cdots\varepsilon_{\nu+1}^2).\end{aligned}$$

4.3 估计

关于向量场 X_{F_ν} 的估计如下.

引理 4.1 设 $D_i^\nu=D(r_{\nu+1}+\frac{i}{4}(r_\nu-r_{\nu+1}),\frac{i}{4}s_\nu)$, 其中 $0<i\leqslant 4$, 则有

$$\|X_{F_\nu}\|_{D_3^\nu,\mathcal{O}_\nu}^a\leqslant\gamma_\nu^{-2}\mathrm{e}^{\mathrm{ce}^{\sqrt{\tau_\nu}}}(r_\nu-r_{\nu+1})^{-2}\varepsilon_0^4\prod_{i=0}^{\nu}\varepsilon_i^2. \tag{4.5}$$

证明 由 (4.4) 及加权范数的定义, 可得

$$|F_{\nu kl\bar\alpha\bar\beta\tilde\alpha\tilde\beta}|_{\mathcal{O}_\nu}\leqslant\gamma_\nu^{-2}\prod_{i_j\in S_{b+\nu}}(1+|k_j|^2j^2(s))^{2\tau_\nu}|k||\grave{P}_{\nu kl\bar\alpha\bar\beta\tilde\alpha\tilde\beta}|_{\mathcal{O}_\nu}.$$

由此可推导出

$$\begin{aligned}&\frac{1}{s^2}\|F_{\nu_{\theta(\nu)}}\|_{\mathcal{D}_3,\mathcal{O}_\nu}\\ &\leqslant\frac{1}{s^2}\sum_{\substack{2|l|+|\bar\alpha+\bar\beta|\leqslant 4\\ 2|l|+|\bar\alpha+\bar\beta|\leqslant 3\\ |\bar\alpha+\bar\beta|+|\tilde\alpha+\tilde\beta|\leqslant 4}}\sum_{\substack{k\\ |k|+|\bar\alpha-\bar\beta|+|\tilde\alpha-\tilde\beta|\neq 0}}|F_{\nu kl\bar\alpha\bar\beta\tilde\alpha\tilde\beta}|_{\mathcal{O}_\nu}|k|\mathrm{e}^{|k|(r_\nu-\frac14(r_\nu-r_{\nu+1}))}|I(v)|^l|z|^{\bar\alpha}|\bar z|^{\bar\beta}|q|^{\tilde\alpha}|\bar q|^{\tilde\beta}\\ &\leqslant\frac{1}{s^2}\sum_{\substack{2|l|+|\bar\alpha+\bar\beta|\leqslant 4\\ 2|l|+|\bar\alpha+\bar\beta|\leqslant 3\\ |\bar\alpha+\bar\beta|+|\tilde\alpha+\tilde\beta|\leqslant 4}}\sum_k\gamma_\nu^{-2}\prod_{i_j\in S_{b+\nu}}(1+|k_j|^2j^2(s))^{2\tau_\nu}|k|^2\mathrm{e}^{-\frac14|k|(r_\nu-r_{\nu+1})}\\ &\quad\times|\grave{P}_{\nu kl\bar\alpha\bar\beta\tilde\alpha\tilde\beta}|_{\mathcal{O}_\nu}\mathrm{e}^{|k|r_\nu}|I(\nu)|^l|z|^{\bar\alpha}|\bar z|^{\bar\beta}|q|^{\tilde\alpha}|\bar q|^{\tilde\beta}\\ &\leqslant\gamma_\nu^{-2}\left(\frac{8\tau_\nu+8}{(r_\nu-r_{\nu+1})\mathrm{e}}\right)^2\mathrm{e}^{\mathrm{ce}^{\sqrt{\tau_\nu}}}\|X_{R_\nu}\|^{\bar a}\end{aligned} \tag{4.6}$$

$$\leqslant \gamma_\nu^{-2}\left(\frac{8\tau_\nu+8}{(r_\nu-r_{\nu+1})\mathrm{e}}\right)^2 \mathrm{e}^{\mathrm{ce}^{\sqrt[4]{\tau_\nu}}}\varepsilon_0^4\prod_{i=0}^{\nu}\varepsilon_i^2.$$

注 4.1 不等式 (4.6) 的推导可分两种情况: 当 $|l|=0$ 时, 通过初等计算知算子特征值的代数和有一次函数量级的下界, 通过对所估计项求导得出最大值, 则有

$$\frac{1}{s^2}\|F_{\nu_{\theta(\nu)}}\|_{\mathcal{D}_3,\mathcal{O}_\nu}\leqslant\gamma_\nu^{-2}\left(\frac{8\tau_\nu+8}{(r_\nu-r_{\nu+1})\mathrm{e}}\right)^2\|X_{R_\nu}\|^{\bar{a}};$$

当 $|l|\neq 0$ 时, 有 $|\bar{\alpha}+\bar{\beta}|+|\tilde{\alpha}+\tilde{\beta}|\leqslant 2$, 借鉴文献 [10, 命题 7.1] 的证明思路, 可验证

$$\frac{1}{s^2}\|F_{\nu_{\theta(\nu)}}\|_{\mathcal{D}_3,\mathcal{O}_\nu}\leqslant\gamma_\nu^{-2}\left(\frac{8\tau_\nu+8}{(r_\nu-r_{\nu+1})\mathrm{e}}\right)^2\mathrm{e}^{\mathrm{ce}^{\sqrt[4]{\tau_\nu}}}\|X_{R_\nu}\|^{\bar{a}}.$$

同理有

$$\begin{aligned}
&\|F_{\nu_{I(\nu)}}\|_{\mathcal{D}_3,\mathcal{O}_\nu}\leqslant\gamma_\nu^{-2}\left(\frac{8\tau_\nu+8}{(r_\nu-r_{\nu+1})\mathrm{e}}\right)^2\mathrm{e}^{\mathrm{ce}^{\sqrt[4]{\tau_\nu}}}\varepsilon_0^4\prod_{i=0}^{\nu}\varepsilon_i^2,\\
&\frac{1}{s}\sum_{n=i_{b+\nu+1}}\|F_{\nu_{z_n}}\|_{\mathcal{D}_3,\mathcal{O}_\nu}|n|^a\mathrm{e}^{|n|\varrho}\leqslant\gamma_\nu^{-2}\left(\frac{8\tau_\nu+8}{(r_\nu-r_{\nu+1})\mathrm{e}}\right)^2\mathrm{e}^{\mathrm{ce}^{\sqrt[4]{\tau_\nu}}}\varepsilon_0^4\prod_{i=0}^{\nu}\varepsilon_i^2,\\
&\frac{1}{s}\sum_{n=i_{b+\nu+1}}\|F_{\nu_{\bar{z}_n}}\|_{\mathcal{D}_3,\mathcal{O}_\nu}|n|^a\mathrm{e}^{|n|\varrho}\leqslant\gamma_\nu^{-2}\left(\frac{8\tau_\nu+8}{(r_\nu-r_{\nu+1})\mathrm{e}}\right)^2\mathrm{e}^{\mathrm{ce}^{\sqrt[4]{\tau_\nu}}}\varepsilon_0^4\prod_{i=0}^{\nu}\varepsilon_i^2,\\
&\frac{1}{s}\sum_{n\in\mathbb{Z}_1^{b+\nu+1}}\|F_{\nu_{q_n}}\|_{\mathcal{D}_3,\mathcal{O}_\nu}|n|^a\mathrm{e}^{|n|\varrho}\leqslant\gamma_\nu^{-2}\left(\frac{8\tau_\nu+8}{(r_\nu-r_{\nu+1})\mathrm{e}}\right)^2\mathrm{e}^{\mathrm{ce}^{\sqrt[4]{\tau_\nu}}}\varepsilon_0^4\prod_{i=0}^{\nu}\varepsilon_i^2,\\
&\frac{1}{s}\sum_{n\in\mathbb{Z}_1^{b+\nu+1}}\|F_{\nu_{\bar{q}_n}}\|_{\mathcal{D}_3,\mathcal{O}_\nu}|n|^a\mathrm{e}^{|n|\varrho}\leqslant\gamma_\nu^{-2}\left(\frac{8\tau_\nu+8}{(r_\nu-r_{\nu+1})\mathrm{e}}\right)^2\mathrm{e}^{\mathrm{ce}^{\sqrt[4]{\tau_\nu}}}\varepsilon_0^4\prod_{i=0}^{\nu}\varepsilon_i^2.
\end{aligned}$$

最后得到

$$\|X_{F_\nu}\|^a_{D_3^\nu,\mathcal{O}_\nu}\leqslant\gamma_\nu^{-2}\mathrm{e}^{\mathrm{ce}^{\sqrt[4]{\tau_\nu}}}(r_\nu-r_{\nu+1})^{-2}\varepsilon_0^4\prod_{i=0}^{\nu}\varepsilon_i^2.$$

证毕. □

接着给出流映射 $\Phi^t_{F_\nu}$ 的估计. (4.7) 用于说明坐标变换是适定的, 而 (4.8) 将用于证明迭代的收敛性.

根据选择的参数 $\gamma_\nu=(\prod_{i=0}^{\nu-1}\varepsilon_i)\varepsilon_\nu^{\frac{3}{4}}$, $\tau_\nu=25(\nu+b+1)$, $r_{\nu+1}=\frac{r_\nu}{2}+\frac{r_0}{4}$ 和 $\varepsilon_\nu=\varepsilon_{\nu-1}^{\frac{9}{5}}$, 可以得到如下引理.

引理 4.2 设 $\eta=(\varepsilon_0^4\prod_{i=0}^{\nu}\varepsilon_i^2)^{\frac{1}{3}}$, $D^\nu_{i\eta}=D(r_{\nu+1}+\frac{i}{4}(r_\nu-r_{\nu+1}),\frac{i}{4}\eta s_\nu)$ (其中 $0<i\leqslant 4$), 当 $\varepsilon\ll\frac{1}{2}\gamma_\nu^3(r_\nu-r_{\nu+1})^3$ 时, 有

$$\Phi^t_{F_\nu}:D^\nu_{2\eta}\to D^\nu_{3\eta},\quad -1\leqslant t\leqslant 1 \tag{4.7}$$

且

$$\|D\Phi^t_{F_\nu}-\mathrm{Id}\|_{D^\nu_{1\eta}}\leqslant\varepsilon_\nu^{\frac{1}{2}}. \tag{4.8}$$

本引理的证明可参见文献 [22].

注 4.2 由于技术限制, 本文的每个 KAM 步骤需要多次重复上述流程. 具体而言, 取 $\varepsilon_{\nu+1}=\varepsilon_{\nu}^{\frac{9}{5}}$, 并且对新的 Hamilton 量作尺度变换 $\varepsilon_{\nu+1}^{18}$, 在重新赋范后, 可通过数学归纳法证明: 新扰动会包含更小的因子 $\varepsilon_{\nu}^{2}\varepsilon_{\nu+1}^{2}$, 且满足

$$\varepsilon_{v}^{2}\varepsilon_{v+1}^{2}\varepsilon_{v+1}^{18}=\varepsilon_{v}^{38}=(\varepsilon_{v}^{\frac{1}{2}})^{76}.$$

因此, 在每个 KAM 步骤中, 需要将上述流程重复 76 次. 最终, 新扰动满足

$$\|X_{\acute{P}_{\nu+1}}\|_{D_{\rho_{\nu+1}}(r_{\nu+1},s_{\nu+1}),\mathcal{O}_{\nu+1}}^{\bar{a}}\leqslant\prod_{i=0}^{\nu+1}\varepsilon_{i}^{2},\quad \|X_{\grave{P}_{\nu+1}}\|_{D_{\rho_{\nu+1}}(r_{\nu+1},s_{\nu+1}),\mathcal{O}_{\nu+1}}^{\bar{a}}\leqslant\varepsilon_{0}^{4}\prod_{i=0}^{\nu+1}\varepsilon_{i}^{2}.$$

5 迭代引理、收敛性与测度估计

给定任意的 $s,r_0,\varepsilon_0,\xi(0)=\xi_0$, 对于 $\nu\geqslant 1$, 定义

$$r_{\nu+1}=r\left(1-\sum_{i=2}^{\nu+2}2^{-i}\right),$$

$$\varepsilon_{\nu+1}=\varepsilon_{\nu}^{\frac{9}{5}},$$

$$\gamma_{\nu}=\left(\prod_{i=0}^{\nu-1}\varepsilon_{i}\right)\varepsilon_{\nu}^{\frac{3}{4}},\quad \tau_{\nu}=25(\nu+b+1),$$

$$\xi(\nu)=(\xi_n)_{n=i_{b+\nu}},\quad I(\nu)=(I_n)_{n\in S_{b+\nu}},\quad \theta(\nu)=(\theta_n)_{n\in S_{b+\nu}},$$

$$\omega^{\nu}(\xi(0),\ldots,\xi(\nu))=(\omega_0(\xi(0),\ldots,\xi(\nu)),\ldots,\omega_{\nu}(\xi(0),\ldots,\xi(\nu))),$$

$$\bar{\Omega}^{\nu}=(\Omega_n^{\nu})_{n=i_{b+\nu+1}},$$

$$\tilde{\Omega}^{\nu}=(\Omega_n^{\nu})_{n\in\mathbb{Z}_1^{b+\nu+1}},$$

$$\mathcal{O}_{\nu}=$$

$$\left\{(\xi(0),\ldots,\xi(\nu))\in\mathcal{O}_{\nu-1}\times[1,2]\,\middle|\,\begin{array}{l}|\langle k,\omega^{\nu}(\xi(0),\ldots,\xi(\nu))\rangle+\langle d,\bar{\Omega}^{\nu}\rangle+\langle l,\tilde{\Omega}^{\nu}\rangle|\geqslant\dfrac{\gamma_{\nu}}{\prod_{i_j\in S_{b+\nu}}(1+|k_j|^2j^2(s))^{\tau_{\nu}}},\\ |k|+|d|+|l|\neq 0,|d|+|l|\leqslant 4,|l|\leqslant 3\end{array}\right\}.$$

5.1 迭代引理

总结上述迭代分析, 得到以下引理.

引理 5.1 若 ε_0 充分小且 $\nu\geqslant 1$, 则 Hamilton 函数 $H_{\nu}=N_{\nu}+P_{\nu}=N_{\nu}+\acute{P}_{\nu}+\grave{P}_{\nu}$ 关于 $(\theta(\nu),I(\nu),z,\bar{z})\in D_{\nu}\in D_{\rho_{\nu}}(r_{\nu},s_{\nu})$ 实解析, 且关于 $(\xi(0),\xi(1),\ldots,\xi(\nu))\in\mathcal{O}_{\nu}\subset[1,2]^{b+\nu}$ 在 Whitney 意义下光滑, 同时满足 $\sum_{i_j\in S_{b+\nu}}k_ji_j+\sum_{n=i_{b+\nu+1}}d_nn+\sum_{n\in\mathbb{Z}_1^{b+\nu+1}}l_nn=0$; 此外, 还满足

$$\|X_{\acute{P}_{\nu}}\|_{D_{\rho_{\nu}}(r_{\nu},s_{\nu}),\mathcal{O}_{\nu}}^{\bar{a}}\leqslant\prod_{i=0}^{\nu}\varepsilon_{i}^{2},\quad \|X_{\grave{P}_{\nu}}\|_{D_{\rho_{\nu}}(r_{\nu},s_{\nu}),\mathcal{O}_{\nu}}^{\bar{a}}\leqslant\varepsilon_{0}^{4}\prod_{i=0}^{\nu}\varepsilon_{i}^{2}.$$

存在参数集

$$\mathcal{O}_{\nu+1}=$$

$$\left\{(\xi(0),\ldots,\xi(\nu+1))\in\mathcal{O}_{\nu}\times[1,2]\,\middle|\,\begin{array}{l}|\langle k,\omega^{\nu+1}\rangle+\langle d,\bar{\Omega}^{\nu+1}\rangle+\langle l,\tilde{\Omega}^{\nu+1}\rangle|\geqslant\dfrac{\gamma_{\nu+1}}{\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))^{\tau_{\nu+1}}},\\ |k|+|d|+|l|\neq 0,|d|+|l|\leqslant 4,|l|\leqslant 3\end{array}\right\},$$

可以构造辛变换

$$\Phi_\nu : D_{\rho_\nu}(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_\nu \to D_{\rho_\nu}(r_\nu, s_\nu), \tag{5.1}$$

使得其在 $D_{\rho_{\nu+1}}(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_{\nu+1}$ 上满足 $H_{\nu+1} = H_\nu \circ \Phi_\nu$, 且 $H_{\nu+1}$ 具有形式

$$H_{\nu+1} = N_{\nu+1} + P_{\nu+1} = N_{\nu+1} + \acute{P}_{\nu+1} + \grave{P}_{\nu+1}.$$

该函数关于 $(\theta(\nu+1), I(\nu+1), z, \bar{z}) \in D_{\nu+1} = D_{\rho_{\nu+1}}(r_{\nu+1}, s_{\nu+1})$ 实解析, 关于 $(\xi(0), \xi(1), \ldots, \xi(\nu+1)) \in \mathcal{O}_{\nu+1} \subset [1,2]^{b+\nu+1}$ 在 Whitney 意义下光滑, 同时满足 $\sum_{i_j \in S_{b+\nu+1}} k_j i_j + \sum_{n=i_{b+\nu+2}} d_n n + \sum_{n \in \mathbb{Z}_1^{b+\nu+2}} l_n n = 0$, 其中

$$\begin{aligned}
&N_{\nu+1} = \langle \omega^\nu(\xi(0), \ldots, \xi(\nu+1)), I(\nu+1) \rangle + \sum_{n \in \mathbb{Z}_1^{b+\nu+1}} \Omega_n^{\nu+1}(\xi(0), \ldots, \xi(\nu+1)) z_n \bar{z}_n,\\
&\acute{P}_{\nu+1} = -\sum_{i_j \in S_{b+\nu+1}} \frac{\varepsilon_0^2 \varepsilon_1^2 \cdots \varepsilon_{\nu+1}^2}{4\pi} I_j^2 - \sum_{n \in \mathbb{Z}_1^{b+\nu+1}} \frac{\varepsilon_0^2 \varepsilon_1^2 \cdots \varepsilon_{\nu+1}^2}{4\pi} |z_n|^4,\\
&\grave{P}_{\nu+1} = O(\varepsilon_0^6 \varepsilon_1^2 \varepsilon_2^2 \cdots \varepsilon_{\nu+1}^2),\\
&\|X_{\acute{P}_{\nu+1}}\|^{\bar{a}}_{D_{\rho_{\nu+1}}(r_{\nu+1}, s_{\nu+1}), \mathcal{O}_{\nu+1}} \leqslant \prod_{i=0}^{\nu+1} \varepsilon_i^2, \quad \|X_{\grave{P}_{\nu+1}}\|^{\bar{a}}_{D_{\rho_{\nu+1}}(r_{\nu+1}, s_{\nu+1}), \mathcal{O}_{\nu+1}} \leqslant \varepsilon_0^4 \prod_{i=0}^{\nu+1} \varepsilon_i^2.
\end{aligned}$$

5.2 收敛性

当 $\nu = 0$ 时, 迭代引理 (引理 5.1) 的假设成立. 令

$$\begin{aligned}
&\varepsilon_0 = \varepsilon, \quad r_0 = r, \quad s_0 = s, \quad L_0 = L, \quad N_0 = N, \quad P_0 = P,\\
&\mathcal{O}_0 = \left\{ (\xi(0)) \in [1,2]^b \;\middle|\; \begin{array}{l} |\langle k, \omega^0(\xi(0)) \rangle + \langle d, \bar{\Omega}^0 \rangle + \langle l, \tilde{\Omega}^0 \rangle| \geqslant \dfrac{\gamma_0}{\prod_{i_j \in S_b} (1 + |k_j|^2 j^2(s))^{\tau_0}},\\ |k| + |d| + |l| \neq 0, |d| + |l| \leqslant 4, |l| \leqslant 3 \end{array} \right\}.
\end{aligned}$$

若 ε_0 和 γ 充分小, 则迭代引理的假设成立. 持续迭代可得如下序列:

$$\begin{aligned}
&\mathcal{O}_{\nu+1} \subset \mathcal{O}_\nu \times [1,2],\\
&\Psi^\nu = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_\nu : D_{\rho_\nu}(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_\nu \to D_{\rho_0}(r_0, s_0), \quad \nu \geqslant 0,\\
&H \circ \Psi^\nu = H_{\nu+1} = N_{\nu+1} + P_{\nu+1} = N_{\nu+1} + \acute{P}_{\nu+1} + \grave{P}_{\nu+1}.
\end{aligned}$$

令 $\tilde{\mathcal{O}} = \lim_{\nu \to \infty} \mathcal{O}_\nu$. 如文献 [35] 所示, 引理 4.2 表明 $N_\nu, \Psi^\nu, D\Psi^\nu$ 和 ω_ν 在 $D_{\frac{1}{2}\rho}(\frac{1}{2}r, 0) \times \tilde{\mathcal{O}}$ 上一致收敛, 且有

$$N_\infty = e_\infty + \langle \omega_\infty, I(\infty) \rangle.$$

由于 $\varepsilon_{\nu+1} = \varepsilon_\nu^{\frac{9}{5}} = (\varepsilon_0)^{(\frac{9}{5})^{\nu+1}}$, 所以当 ε 充分小时, $\varepsilon_{\nu+1} \to 0$. 此外, $\sum_{\nu=0}^\infty \varepsilon_\nu \leqslant 2\varepsilon$.

设 ϕ_H^t 是向量场 X_H 对应的流映射. 由于 $H \circ \Psi^\nu = H_{\nu+1}$, 所以有

$$\phi_H^t \circ \Psi^\nu = \Psi^\nu \circ \phi_{H_{\nu+1}}^t. \tag{5.2}$$

$\Psi^\nu, D\Psi^\nu, \omega_\nu$ 与 X_{H_ν} 的一致收敛性保证了可以对 (5.2) 两端取极限. 在 $D_{\frac{1}{2}\rho}(\frac{1}{2}r, 0) \times \tilde{\mathcal{O}}$ 上, 可得

$$\phi_H^t \circ \Psi^\infty = \Psi^\infty \circ \phi_{H_\infty}^t \tag{5.3}$$

且

$$\Psi^\infty : D_{\frac{1}{2}\rho}\left(\frac{1}{2}r, 0\right) \times \tilde{\mathcal{O}} \to D_\rho(r,s) \times \mathcal{O}.$$

由 (5.3) 可知, 对 $\xi \in \tilde{\mathcal{O}}$, 有

$$\phi_H^t(\Psi^\infty(\mathbb{T}^\infty \times \{\xi\})) = \Psi^\infty \phi_{N_\infty}^t(\mathbb{T}^\infty \times \{\xi\}) = \Psi^\infty(\mathbb{T}^\infty \times \{\xi\}).$$

这表明 $\Psi^\infty(\mathbb{T}^\infty \times \{\xi\})$ 是一个嵌入环面, 且当 $\xi \in \tilde{\mathcal{O}}$ 时, 它也是原扰动 Hamilton 系统的不变环面. 与 $\Psi^\infty(\mathbb{T}^\infty \times \{\xi\})$ 关联的频率 $\omega_\infty(\xi)$ 是 $\omega(\xi)$ 的微小修正, 这证明了概周期解的存在性.

5.3 测度估计

根据引理 5.1, 在第 ν 步 KAM 迭代中, 我们需要在以下条件下剔除参数集 $\mathcal{R}_{\nu+1}$:

$$\sum_{i_j \in S_{b+\nu+1}} k_j i_j + \sum_{n=i_{b+\nu+2}} d_n n + \sum_{n \in \mathbb{Z}_1^{b+\nu+2}} l_n n = 0,$$

其中

$$\mathcal{R}_{\nu+1} = \left\{ (\xi(0), \dots, \xi(\nu+1)) \in \mathcal{O}_\nu \times [1,2] \,\middle|\, \begin{array}{l} |\langle k, \omega^{\nu+1}\rangle + \langle d, \bar{\Omega}^{\nu+1}\rangle + \langle l, \tilde{\Omega}^{\nu+1}\rangle| < \dfrac{\gamma_{\nu+1}}{\prod_{i_j \in S_{b+\nu+1}} (1+|k_j|^2 j^2(s))^{\tau_{\nu+1}}}, \\ |k| + |d| + |l| \neq 0, |d| + |l| \leqslant 4, |l| \leqslant 3 \end{array} \right\}.$$

引理 5.2 若 $k = 0$, 则 $\mathcal{R}_{\nu+1} = \emptyset$.

证明 当 $k = 0$, $0 < |d| + |l| \leqslant 4$ 时, 由紧致形式知

$$\sum_{n=i_{b+\nu+2}} d_n n + \sum_{n \in \mathbb{Z}_1^{b+\nu+2}} l_n n = 0,$$

我们分析 $|\langle d, \bar{\Omega}^{\nu+1}\rangle + \langle l, \tilde{\Omega}^{\nu+1}\rangle|$, 按 (d,l) 其中负号的数量, 分为以下五种情形.

情形 1 无负号. 计算小分母有

$$|\langle d, \bar{\Omega}^{\nu+1}\rangle + \langle l, \tilde{\Omega}^{\nu+1}\rangle| \geqslant \frac{1}{5}\varepsilon_0^{-8} \cdots \varepsilon_{\nu+1}^{-8} - O(\varepsilon_0^6 \varepsilon_1^{-8} \cdots \varepsilon_{\nu+1}^{-8}) \geqslant \frac{1}{6}\varepsilon_0^{-8} \cdots \varepsilon_{\nu+1}^{-8}.$$

情形 2 1 个负号. 结合条件 $\sum_{n=i_{b+\nu+2}} d_n n + \sum_{n \in \mathbb{Z}_1^{b+\nu+2}} l_n n = 0$, 同理可得

$$|\langle d, \bar{\Omega}^{\nu+1}\rangle + \langle l, \tilde{\Omega}^{\nu+1}\rangle| \geqslant \frac{1}{5}\varepsilon_0^{-8} \cdots \varepsilon_{\nu+1}^{-8} - O(\varepsilon_0^6 \varepsilon_1^{-8} \cdots \varepsilon_{\nu+1}^{-8}) \geqslant \frac{1}{6}\varepsilon_0^{-8} \cdots \varepsilon_{\nu+1}^{-8}.$$

情形 3 2 个负号.

- 若 $0 < |d| + |l| < 4$, 处理方式同情形 1 和 2.
- 若 $|d| + |l| = 4$, 结合条件 $\sum_{n=i_{b+\nu+2}} d_n n + \sum_{n \in \mathbb{Z}_1^{b+\nu+2}} l_n n = 0$, 可得

$$|\langle d, \bar{\Omega}^{\nu+1}\rangle + \langle l, \tilde{\Omega}^{\nu+1}\rangle| \geqslant \varepsilon_0^{-8} \cdots \varepsilon_{\nu+1}^{-8} - O(\varepsilon_0^6 \varepsilon_1^{-8} \cdots \varepsilon_{\nu+1}^{-8}) \geqslant \frac{1}{2}\varepsilon_0^{-8} \cdots \varepsilon_{\nu+1}^{-8}.$$

情形 4 3 个负号. 此时正号数量至多为 1, 结合条件 $\sum_{n=i_{b+\nu+2}} d_n n + \sum_{n \in \mathbb{Z}_1^{b+\nu+2}} l_n n = 0$, 可得

$$|\langle d, \bar{\Omega}^{\nu+1}\rangle + \langle l, \tilde{\Omega}^{\nu+1}\rangle| \geqslant \frac{1}{5}\varepsilon_0^{-8} \cdots \varepsilon_{\nu+1}^{-8} - O(\varepsilon_0^6 \varepsilon_1^{-8} \cdots \varepsilon_{\nu+1}^{-8}) \geqslant \frac{1}{6}\varepsilon_0^{-8} \cdots \varepsilon_{\nu+1}^{-8}.$$

情形 5 4 个负号. 计算小分母有

$$|\langle d,\bar{\Omega}^{\nu+1}\rangle+\langle l,\tilde{\Omega}^{\nu+1}\rangle|\geqslant\frac{1}{5}\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}-O(\varepsilon_0^6\varepsilon_1^{-8}\cdots\varepsilon_{\nu+1}^{-8})\geqslant\frac{1}{6}\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}.$$

综上, 引理 5.2 得证. □

引理 5.3 若 $k\neq 0$, $|l|=1$, 且 $|\langle l,\tilde{\Omega}^{\nu+1}\rangle|=|\Omega_n^{\nu+1}|\geqslant\frac{1}{2}|n|^2\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}$, 则当 $|n|>100s(b+\nu+2)\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))$ 时, $\mathcal{R}_{\nu+1}=\emptyset$.

证明 计算小分母有

$$\begin{aligned}&|\langle k,\omega^{\nu+1}\rangle+\langle d,\bar{\Omega}^{\nu+1}\rangle+\langle l,\tilde{\Omega}^{\nu+1}\rangle|\\&\quad\geqslant\frac{1}{2}|n|^2\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}-6(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}\\&\quad\geqslant(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}.\end{aligned}$$

因此, 引理 5.3 得证. □

引理 5.4 若 $k\neq 0$, $|l|=2$, 且 $|\langle l,\tilde{\Omega}^{\nu+1}\rangle|=|\Omega_n^{\nu+1}\pm\Omega_m^{\nu+1}|$, 则当 $\max\{|n|,|m|\}>100(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))$ 时, $\mathcal{R}_{\nu+1}=\emptyset$.

证明 当 $n=-m$ 且 $|\langle l,\tilde{\Omega}^{\nu+1}\rangle|=|\Omega_n^{\nu+1}-\Omega_m^{\nu+1}|$ 时, 条件 $\max\{|n|,|m|\}>100(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))$ 意味着

$$\Big|\sum_{i_j\in S_{b+\nu+1}}k_ji_j+\sum_{n=i_{b+\nu+2}}d_nn+\sum_{n\in\mathbb{Z}_1^{b+\nu+2}}l_nn\Big|\geqslant 100(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))\neq 0.$$

故 $\mathcal{R}_{\nu+1}=\emptyset$.

当 $|n|\neq|m|$ 时,

$$\begin{aligned}|\langle k,\omega^{\nu+1}\rangle+\langle d,\bar{\Omega}^{\nu+1}\rangle+\langle l,\tilde{\Omega}^{\nu+1}\rangle|&\geqslant\frac{1}{2}\max\{|n|,|m|\}\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}\\&\quad-6(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}\\&\geqslant(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}.\end{aligned}$$

因此, 引理 5.4 得证. □

引理 5.5 若 $k\neq 0$, $|l|=3$, 且 $|\langle l,\tilde{\Omega}^{\nu+1}\rangle|=|\Omega_p^{\nu+1}\pm\Omega_m^{\nu+1}\pm\Omega_n^{\nu+1}|$, 则当 $\max\{|p|,|m|,|n|\}>4(100(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s)))^2$ 时, $\mathcal{R}_{\nu+1}=\emptyset$.

证明 不妨设 $|p|\leqslant|m|\leqslant|n|$. 若 $\Omega_m^{\nu+1}$ 与 $\Omega_n^{\nu+1}$ 符号相同, 计算

$$\begin{aligned}|\langle k,\omega^{\nu+1}\rangle+\langle d,\bar{\Omega}^{\nu+1}\rangle+\langle l,\tilde{\Omega}^{\nu+1}\rangle|&=|\langle k,\omega^{\nu+1}\rangle+\langle d,\bar{\Omega}^{\nu+1}\rangle\pm\Omega_p^{\nu+1}\pm(\Omega_m^{\nu+1}+\Omega_n^{\nu+1})|\\&\geqslant\frac{1}{2}\Big(n^2+\frac{1}{5}\Big)\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}\\&\quad-6(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}\end{aligned}$$

$$\geqslant (s(b+\nu+2))^2 \prod_{i_j\in S_{b+\nu+1}} (1+|k_j|^2 j^2(s))\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}.$$

当 $\Omega_m^{\nu+1}$ 与 $\Omega_n^{\nu+1}$ 符号相反时, 分以下三种情形讨论.

情形 1 $|p|\leqslant 100(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))$ 且 $n=-m$. 估计

$$\begin{aligned}\Big|\sum_{i_j\in S_{b+\nu+1}} k_j i_j + \sum_{n=i_{b+\nu+2}} d_n n \pm p \pm (n-m)\Big| &\geqslant |n|-|p|-6s(b+\nu+2)\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))\\ &\geqslant \Big(100(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))\Big)^2\\ &\quad -100(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))\\ &\quad -6s(b+\nu+2)\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))\\ &\neq 0,\end{aligned}$$

故 $\mathcal{R}_{\nu+1}=\emptyset$.

情形 2 $|p|\leqslant 100(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))$, $|n|\neq|m|$ 且 $|n|>4(100(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s)))^2$. 估计

$$\begin{aligned}|\langle k,\omega^{\nu+1}\rangle+\langle d,\bar\Omega^{\nu+1}\rangle+\langle l,\tilde\Omega^{\nu+1}\rangle| &\geqslant \frac{1}{2}|n|\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}-|p|^2\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}\\ &\quad -6(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}\\ &\geqslant (s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}.\end{aligned}$$

情形 3 $|p|>100(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))$. 令 $n=m\pm p+\sum_{i_j\in S_{b+\nu+1}}k_ji_j+\sum_{n=i_{b+\nu+2}}d_nn$, 代入 $-\Omega_n^{\nu+1}+\Omega_m^{\nu+1}\pm\Omega_p^{\nu+1}+\langle k,\omega^{\nu+1}\rangle+\langle d,\bar\Omega^{\nu+1}\rangle$, 可得

$$\begin{aligned}&|-\Omega_n^{\nu+1}+\Omega_m^{\nu+1}\pm\Omega_p^{\nu+1}+\langle k,\omega^{\nu+1}\rangle+\langle d,\bar\Omega^{\nu+1}\rangle|\\ &\quad\geqslant |m||p|\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}-6(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))(|m|+|p|)\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}\\ &\quad\geqslant \Big(|p|-12(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))\Big)|m|\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}\\ &\quad\geqslant (s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}.\end{aligned}$$

因此, 引理 5.5 得证. □

引理 5.6 给定 $k\neq 0,d,l$, 则有

$$\mathrm{meas}(\mathcal{R}_{\nu+1}^{k,d,l})\leqslant\frac{\varepsilon_{\nu+1}^{\frac{1}{18}}}{\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2j^2(s))^{\tau_{\nu+1}}}.$$

证明 首先回顾

$$\xi(0)=\xi,\quad \xi(1)=(\xi_n)_{n=i_{b+1}},\quad \ldots,\quad \xi(\nu+1)=(\xi_n)_{n=i_{b+\nu+1}},$$

$$I(\nu+1)=(I_n)_{n\in S_{b+\nu+1}},\quad \theta(\nu+1)=(\theta_n)_{n\in S_{b+\nu+1}},$$

$$\omega^{\nu+1}(\xi(0),\ldots,\xi(\nu+1))=(\omega_0(\xi(0),\ldots,\xi(\nu+1)),\ldots,\omega_{\nu+1}(\xi(0),\ldots,\xi(\nu+1))),$$

$$\begin{aligned}\omega_{0j}(\xi(0),\ldots,\xi(\nu+1))=\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}\left(|i_j|^2+\frac{1}{5}\right)-\frac{\varepsilon_1^{-8}\cdots\varepsilon_{\nu+1}^{-8}}{2\pi}\xi_j\\ +O(\varepsilon_0^6\varepsilon_1^{-8}\cdots\varepsilon_{\nu+1}^{-8})f_0^{\nu+1}(\xi(0))+O(\varepsilon_0^6\varepsilon_2^{-8}\cdots\varepsilon_{\nu+1}^{-8})f_1^{\nu+1}(\xi(1))\\ +\cdots+O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_{i-1}^2\varepsilon_{i+1}^{-8}\cdots\varepsilon_{\nu+1}^{-8})f_i^{\nu+1}(\xi(i))\\ +\cdots+O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_\nu^2)f_{\nu+1}^{\nu+1}(\xi(\nu+1)),\quad i_j\in S_b,\end{aligned}$$

$$\begin{aligned}\omega_n(\xi(0),\ldots,\xi(\nu+1))=\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}\left(|i_{b+n}|^2+\frac{1}{5}\right)-\frac{\varepsilon_0^2\varepsilon_1^2\cdots\varepsilon_{n-1}^2\varepsilon_{n+1}^{-8}\cdots\varepsilon_{\nu+1}^{-8}}{2\pi}\xi_n\\ +O(\varepsilon_0^6\varepsilon_1^{-8}\cdots\varepsilon_{\nu+1}^{-8})f_0^{\nu+1}(\xi(0))+O(\varepsilon_0^6\varepsilon_2^{-8}\cdots\varepsilon_{\nu+1}^{-8})f_1^{\nu+1}(\xi(1))\\ +\cdots+O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_{i-1}^2\varepsilon_{i+1}^{-8}\cdots\varepsilon_{\nu+1}^{-8})f_i^{\nu+1}(\xi(i))\\ +\cdots+O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_\nu^2)f_{\nu+1}^{\nu+1}(\xi(\nu+1)),\quad n=1,\ldots,\nu+1,\end{aligned}$$

$$\begin{aligned}\Omega_n^{\nu+1}(\xi(0),\ldots,\xi(\nu+1))=\varepsilon_0^{-8}\cdots\varepsilon_{\nu+1}^{-8}\left(|n|^2+\frac{1}{5}\right)\\ +O(\varepsilon_0^6\varepsilon_1^{-8}\cdots\varepsilon_{\nu+1}^{-8})f_0^{\nu+1}(\xi(0))+O(\varepsilon_0^6\varepsilon_2^{-8}\cdots\varepsilon_{\nu+1}^{-8})f_1^{\nu+1}(\xi(1))\\ +\cdots+O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_{i-1}^2\varepsilon_{i+1}^{-8}\cdots\varepsilon_{\nu+1}^{-8})f_i^{\nu+1}(\xi(i))\\ +\cdots+O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_\nu^2)f_{\nu+1}^{\nu+1}(\xi(\nu+1)),\quad n\in\mathbb{Z}_1^{b+\nu+1},\end{aligned}$$

$$\|f_i^{\nu+1}(\xi(i))\|_{\mathcal{O}_{\nu+1}}=\sup_{\mathcal{O}_{\nu+1}}\left(|f_i^{\nu+1}(\xi(i))|+\left|\frac{\partial f_i^{\nu+1}}{\partial\xi(i)}\right|\right)\leqslant 1,\quad i=0,1,\ldots,\nu+1.$$

考虑偏导数

$$\begin{aligned}\left|\frac{\partial\omega_n}{\partial\xi_n}\right|&\geqslant\frac{\varepsilon_0^2\varepsilon_1^2\cdots\varepsilon_{n-1}^2\varepsilon_{n+1}^{-8}\cdots\varepsilon_{\nu+1}^{-8}}{2\pi}-O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_{n-1}^2\varepsilon_{n+1}^{-8}\cdots\varepsilon_{\nu+1}^{-8})\\ &\geqslant\frac{\varepsilon_0^2\varepsilon_1^2\cdots\varepsilon_{n-1}^2\varepsilon_{n+1}^{-8}\cdots\varepsilon_{\nu+1}^{-8}}{4\pi},\quad 0\leqslant n\leqslant\nu+1.\end{aligned}$$

给定 $k=(\ldots,k_n,\ldots)_{0\leqslant n\leqslant\nu+1}\neq 0$, 记 $|k_n|$ 为 k 的最大分量. 由

$$\begin{aligned}\left|\frac{\partial(\langle k,\omega^{\nu+1}\rangle+\langle d,\bar\Omega^{\nu+1}\rangle+\langle l,\tilde\Omega^{\nu+1}\rangle)}{\partial\xi_n}\right|&\geqslant\frac{|k_n|\varepsilon_0^2\varepsilon_1^2\cdots\varepsilon_{n-1}^2\varepsilon_{n+1}^{-8}\cdots\varepsilon_{\nu+1}^{-8}}{4\pi}\\ &\quad-|k_n|O(\varepsilon_0^6\varepsilon_1^2\cdots\varepsilon_{n-1}^2\varepsilon_{n+1}^{-8}\cdots\varepsilon_{\nu+1}^{-8})\\ &\geqslant\frac{|k_n|\varepsilon_0^2\varepsilon_1^2\cdots\varepsilon_{n-1}^2\varepsilon_{n+1}^{-8}\cdots\varepsilon_{\nu+1}^{-8}}{8\pi},\quad 0\leqslant n\leqslant\nu+1,\end{aligned}$$

进而估计测度

$$\begin{aligned}\operatorname{meas}(\mathcal{R}_{\nu+1}^{k,d,l}) &\leqslant \frac{\gamma_{\nu+1}}{\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2 j^2(s))^{\tau_{\nu+1}}}\frac{1}{\varepsilon_0^2\varepsilon_1^2\cdots\varepsilon_\nu^2}\\ &\leqslant \frac{\prod_{i=0}^{\nu}\varepsilon_i\varepsilon_{\nu+1}^{\frac{3}{4}}}{\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2 j^2(s))^{\tau_{\nu+1}}}\frac{1}{\varepsilon_0^2\varepsilon_1^2\cdots\varepsilon_\nu^2}\\ &\leqslant \frac{\varepsilon_{\nu+1}^{\frac{3}{4}}}{\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2 j^2(s))^{\tau_{\nu+1}}}\frac{1}{\varepsilon_\nu^{\frac{5}{4}}}\\ &\leqslant \frac{\varepsilon_{\nu+1}^{\frac{3}{4}}}{\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2 j^2(s))^{\tau_{\nu+1}}}\frac{1}{\varepsilon_{\nu+1}^{\frac{25}{36}}}\\ &\leqslant \frac{\varepsilon_{\nu+1}^{\frac{1}{18}}}{\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2 j^2(s))^{\tau_{\nu+1}}}.\end{aligned}$$

证毕. □

引理 5.7 在 KAM 迭代过程中, 挖掉的集合总测度满足

$$\operatorname{meas}\Big(\bigcup_{\nu\geqslant 0}\mathcal{R}_{\nu+1}\Big)=\operatorname{meas}\Big(\bigcup_{\nu\geqslant 0}\bigcup_{k,d,l}\mathcal{R}_{\nu+1}^{k,d,l}\Big)\leqslant\varepsilon_0^{\frac{1}{20}}.$$

证明 由引理 5.6 知 $\operatorname{meas}(\mathcal{R}_{\nu+1}^{k,d,l})\leqslant\frac{\varepsilon_{\nu+1}^{\frac{1}{18}}}{\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2 j^2(s))^{\tau_{\nu+1}}}$. 结合引理 5.2–5.5, 可得

$$\begin{aligned}\operatorname{meas}\Big(\bigcup_{\nu\geqslant 0}\mathcal{R}_{\nu+1}\Big)&=\operatorname{meas}\Big(\bigcup_{\nu\geqslant 0}\bigcup_{k,d,l}\mathcal{R}_{\nu+1}^{k,d,l}\Big)\\ &\leqslant\sum_{\nu\geqslant 0}\sum_{k}\frac{[4(100(s(b+\nu+2))^2\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2 j^2(s)))^2]^4\varepsilon_{\nu+1}^{\frac{1}{18}}}{\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2 j^2(s))^{\tau_{\nu+1}}}\\ &\leqslant\sum_{\nu\geqslant 0}\sum_{k}\frac{\varepsilon_{\nu+1}^{\frac{1}{20}}}{\prod_{i_j\in S_{b+\nu+1}}(1+|k_j|^2 j^2(s))^{25(\nu+b)}}\\ &\leqslant\sum_{\nu\geqslant 0}\varepsilon_{\nu+1}^{\frac{1}{20}}\leqslant\varepsilon_0^{\frac{1}{20}}.\end{aligned}$$

至此, 测度估计的证明完成. □

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Almost periodic solutions of one-dimensional nonlinear Schrödinger equations with the constant potential

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Abstract Under periodic boundary conditions, we consider the one-dimensional nonlinear Schrödinger equation with a constant potential m,

$$\mathrm{i}u_t - \Delta u + mu + f(|u|^2)u = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{T},$$

where f is a real analytic function in some neighborhood of the origin satisfying $f(0) = 0$ and $f'(0) \neq 0$. In this paper, based on the analysis of frequency shift, strong Diophantine conditions and an improved infinite-dimensional KAM (Kolmogorov-Arnold-Moser) theory, we establish that, for any given constant potential m, the equation admits a Whitney smooth family of small amplitude, time almost periodic solutions. We emphasize that our solutions are not maximal tori; instead, they are predominantly localized on a sparse admissible set.

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