



三维可压缩等熵流边界层方程的局部适定性

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摘要 Prandtl 方程作为不可压流动中描述边界层行为的基本模型, 在不同背景下已被广泛研究. 然而, 对于可压缩流的边界层问题, 目前的数学研究结果相对较少. 尤其是三维情形, 无论是不可压缩流还是可压缩流边界层问题, 研究结果都很少. 本文考虑具有无滑移边界条件的三维可压缩等熵流边界层方程. 在初值具有切向解析、法向具有 Sobolev 正则性的条件下, 本文利用直接的能量估计建立该系统的局部适定性.

关键词 边界层方程 三维可压缩流 指数权 适定性 能量方法

MSC (2020) 主题分类 35M13, 35Q35, 76N20

1 引言

考虑定义在区域 $\mathbb{R}_+^3 := \{(x, y, z) \in \mathbb{R}^3, z > 0\}$ 上的三维可压缩等熵流边界层方程

$$\begin{cases} \partial_t \mathbf{u}^p + \mathbf{u}^p \cdot \nabla \mathbf{u}^p + w^p \partial_z \mathbf{u}^p + \nabla P = \frac{1}{\rho} \partial_z^2 \mathbf{u}^p, \\ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}^p) + \partial_z (\rho w^p) = 0, \end{cases} \quad (1.1)$$

附加初值条件

$$\mathbf{u}^p(0, x, y, z) = \mathbf{u}_0^p(x, y, z), \quad (1.2)$$

以及边界条件

$$\mathbf{u}^p(t, x, y, z)|_{z=0} = 0, \quad \lim_{z \rightarrow +\infty} \mathbf{u}^p(t, x, y, z) = \mathbf{U}(t, x, y), \quad (1.3)$$

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其中切向梯度算子 $\nabla = (\partial_x, \partial_y)$, 未知函数 $\mathbf{u}^p := (u^p(t, x, y, z), v^p(t, x, y, z))$ 表示速度场的切向分量, 未知函数 $w^p := w^p(t, x, y, z)$ 表示速度场的法向分量. 已知函数 $\rho(t, x, y)$ 表示 Euler 外流的密度在边界上的迹, $\mathbf{U} := (U(t, x, y), V(t, x, y))$ 和 $P(t, x, y)$ 分别表示外流的切向速度场和焓在边界上的迹, 它们满足如下的 Bernoulli 定律:

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} + \nabla P = 0. \quad (1.4)$$

在陈述本文主要结果之前, 首先回顾与本文密切相关的非稳态边界层方程相关的研究进展. 不可压流边界层方程最早由 Prandtl^[28] 于 1904 年提出, 后也被称为 Prandtl 方程. 围绕 Prandtl 方程的研究已有很长的历史, 由于耦合性和退化性的特点, 该方程的数学研究面临诸多挑战.

在切向速度关于法向变量满足单调性条件的前提下, Oleinik^[24] 于 1966 年应用 Crocco 变换首次证明了 Prandtl 方程局部解的存在性, 这一重要结果以及 Oleinik 与合作者的其他相关工作可参见文献^[25]. 大约半个世纪以后, Alexandre 等^[1] 通过引入抵消机制来克服切向导数损失所带来的困难, 利用经典能量估计的方法结合 Nash-Moser 迭代技巧, 得到了 Prandtl 方程在代数加权 Sobolev 空间下的适定性结果. 同一时间, Masmoudi 和 Wong^[23] 也独立发现了这种抵消机制, 通过引入辅助函数, 利用直接的能量估计, 得到了与文献^[1] 类似的结果. 此外, 在单调性假设并结合顺压差条件 ($\partial_x P \leq 0$) 的情形下, Xin 和 Zhang^[36] 证明了 Prandtl 方程整体弱解的存在性. 最近, Xin 等^[37] 进一步证明了此解是经典解. 从物理的观点来看, 单调性假设具有稳定性作用, 一旦这个条件被破坏, 边界层的分离现象可能会出现, 关于 Prandtl 方程在 Sobolev 空间中的不适定性问题, 可参见文献^[7, 9, 11, 13, 14, 35].

在没有单调性假设的条件下, 通过附加初值的解析性条件, 对 Prandtl 方程解的适定性研究最早可追溯至 Sammartino 和 Caffisch 的工作. 他们在文献^[30] 中利用抽象的 Cauchy-Kowalewski 定理证明了二维 Prandtl 方程在解析初值下的局部适定性. 随后, 该结果在文献^[22] 中得到了改进, 降低了对法向解析性的要求. Kukavica 和 Vicol^[16] 引入了新的解析范数, 利用直接的能量方法, 证明了二维 Prandtl 方程在解析框架下的局部适定性. 当初值足够小且有切向解析性时, Zhang 和 Zhang^[39] 证明了对于小外流, Prandtl 方程存在唯一的长时间解. 在此之后, Ignatova 和 Vicol^[15] 通过引入合适的好函数, 并利用抵消思想获得了时间阻尼项, 成功延长了解的生命周期. 在此基础上, 通过对该好函数的改进, 在外流足够小且随时间衰减的条件下, Paicu 和 Zhang^[26] 建立了 Prandtl 方程在解析空间中的整体适定性. 当将解析性要求拓展至 Gevrey 正则性时, 近年来也有一些新的进展, Gérard-Varet 和 Masmoudi^[10] 在初值有唯一的非退化临界点的条件下, 证明了 Prandtl 方程在 $G^{7/4}$ 空间中的适定性. 进一步地, Li 和 Yang^[18] 在非退化临界点假设的前提下, 得到了该方程在临界指标 G^2 下 Gevrey 解的存在唯一性. 随后, 通过构造适当的辅助函数及检验函数, 在没有初值单调性假设或者非退化临界点的结构性假设时, Dietert 和 Gérard-Varet^[6] 得到了 G^2 空间中 Prandtl 方程的局部适定性. 更多关于二维 Prandtl 方程适定性理论, 可参见文献^[2, 3, 31, 32].

当研究从二维拓展至三维情形时, 由于多维速度场带来的复杂边界层结构以及二次流产生, 关于 Prandtl 方程的适定性研究更加困难. 除解析空间结果^[30, 39] 外, Liu 等^[21] 在有限阶光滑函数空间中, 基于单调性假设与一类特定的结构条件, 建立了三维 Prandtl 方程的适定性. 最近, Li 等^[17] 在无需任何结构假设的情形下, 采用新的抵消机制, 在 Gevrey 2 正则性框架下证明了 Prandtl 系统的局部适定性. 其他相关研究可参见文献^[19, 20, 27, 29].

与不可压缩流边界层方程已有的深入研究相比, 可压缩流边界层的研究成果相对较少. Wang 和 Williams^[33] 最早推导了满足 Navier 边界条件的可压缩等熵流边界层方程, 并分析了其稳定性. 在无滑移边界条件下, 基于与经典 Prandtl 方程相同的单调性假设, 研究者分别采用不同的方法, 证明

了二维可压缩等熵流边界层方程在 Sobolev 空间的局部适定性 [8,12,34]. 在不依赖单调性假设的情形下, Chen 等 [5] 针对小解析初值与特定外流, 进一步证明了二维情形下解的整体存在性. 最近, Chen 等 [4] 基于经典的 Cauchy-Kowalewski 定理引入变量变换, 利用能量估计方法, 证明了三维可压缩等熵流边界层方程在带代数权的解析空间中的局部适定性. 关于可压缩流边界层的其他理论研究, 可参见文献 [38].

本文研究三维可压缩等熵流边界层问题 (1.1)–(1.3), 不同于文献 [4] 中引入变量变换的方法, 本文通过引入 Gauss 权函数克服线性增长项 $z\mathcal{U}\partial_z \mathbf{u}$ 带来的困难, 利用直接的能量估计得到该初边值问题的局部适定性. 一方面, 避免了引入变量变换带来的复杂计算; 另一方面, 避免了 Cauchy-Kowalewski 定理的运用, 方法上更为简洁直观.

2 主要结论

首先, 对边界条件 (1.3) 进行齐次化处理, 为此, 引入

$$\mathbf{u}(t, x, y, z) = \mathbf{u}^p(t, x, y, z) - [1 - \phi(z)]\mathbf{U}(t, x, y), \quad w(t, x, y, z) = w^p(t, x, y, z), \quad (2.1)$$

其中 $\phi(z) = \frac{2\sqrt{\lambda}}{\sqrt{\pi}} \int_z^{+\infty} e^{-\lambda\eta^2} d\eta$, 参数 $\lambda > 0$, 其具体范围将在后文给出.

此时, 方程 (1.1)₂ 变形为

$$w = -z\mathcal{U} - \nabla \cdot \mathbf{g} - \nabla \tilde{\rho} \cdot \mathbf{g} + \Phi(\nabla \tilde{\rho} \cdot \mathbf{U} + \nabla \cdot \mathbf{U}), \quad (2.2)$$

其中 $\tilde{\rho} = \log \rho$, $\mathcal{U} = \partial_t \tilde{\rho} + \nabla \cdot \mathbf{U} + \nabla \tilde{\rho} \cdot \mathbf{U}$, $\Phi = \int_0^z \phi(\tilde{z}) d\tilde{z}$ 和 $\mathbf{g} = (\int_0^z u d\tilde{z}, \int_0^z v d\tilde{z})$.

将 (2.1) 和 (2.2) 代入方程 (1.1), 可得方程

$$\partial_t \mathbf{u} - \rho^{-1} \partial_z^2 \mathbf{u} + \mathcal{L}(\mathbf{u}) + \mathcal{N}(\mathbf{u}) = \mathcal{F}, \quad (2.3)$$

其中线性项

$$\begin{aligned} \mathcal{L}(\mathbf{u}) = & -z\mathcal{U}\partial_z \mathbf{u} + (1 - \phi)\mathbf{u} \cdot \nabla \mathbf{U} + (1 - \phi)\mathbf{U} \cdot \nabla \mathbf{u} + \phi'(\nabla \tilde{\rho} \cdot \mathbf{g})\mathbf{U} \\ & + \phi'(\nabla \cdot \mathbf{g})\mathbf{U} + \Phi(\nabla \tilde{\rho} \cdot \mathbf{U} + \nabla \cdot \mathbf{U})\partial_z \mathbf{u}, \end{aligned} \quad (2.4)$$

非线性项

$$\mathcal{N}(\mathbf{u}) = -(\nabla \cdot \mathbf{g})\partial_z \mathbf{u} - (\nabla \tilde{\rho} \cdot \mathbf{g})\partial_z \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}, \quad (2.5)$$

源项

$$\begin{aligned} \mathcal{F} = & -z\mathcal{U}\phi'\mathbf{U} + \phi(1 - \phi)\mathbf{U} \cdot \nabla \mathbf{U} - \rho^{-1}\phi''\mathbf{U} - \phi\nabla P \\ & + \Phi(\nabla \tilde{\rho} \cdot \mathbf{U})\phi'\mathbf{U} + \Phi(\nabla \cdot \mathbf{U})\phi'\mathbf{U}, \end{aligned} \quad (2.6)$$

且未知量 $\mathbf{u}$ 满足初值条件

$$\mathbf{u}(0, x, y, z) = \mathbf{u}_0(x, y, z) = \mathbf{u}_0^p(x, y, z) - (1 - \phi(z))\mathbf{U}_0(x, y), \quad (2.7)$$

以及齐次边界条件

$$\mathbf{u}(t, x, y, 0) = 0, \quad \lim_{z \rightarrow +\infty} \mathbf{u}(t, x, y, z) = 0. \quad (2.8)$$

为陈述主要结果, 引入求解初边值问题 (2.3)–(2.8) (或 (1.1)–(1.3)) 的函数空间. 定义如下的带权实解析函数空间, 其在切向变量 (x, y) 方向上是实解析的, 在法向变量 z 上属于加权的 Sobolev 空间:

$$X_{\tau, \mu} = \{\forall \alpha \in \mathbb{N}^2, \theta_\mu \partial_h^\alpha \mathbf{u}(t, x, y, z) \in L^2(\mathbb{R}_+^3; dx dy dz) : \|\mathbf{u}\|_{X_{\tau, \mu}} < +\infty\}, \quad (2.9)$$

这里 $\partial_h^\alpha = \partial_x^{\alpha_1} \partial_y^{\alpha_2}$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, 并伴有如下范数:

$$\begin{aligned} \|\mathbf{u}\|_{X_{\tau, \mu}}^2 &= \sum_{m \geq 0} X_m^2, & \|\mathbf{u}\|_{Y_{\tau, \mu}}^2 &= \sum_{m \geq 1} Y_m^2, \\ \|\mathbf{u}\|_{Z_{\tau, \mu}}^2 &= \sum_{m \geq 0} Z_m^2, & \|\mathbf{u}\|_{D_{\tau, \mu}}^2 &= \sum_{m \geq 0} D_m^2, \end{aligned} \quad (2.10)$$

其中

$$X_m = \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^m M_m, \quad (2.11)$$

$$Y_m = \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^{m-1/2} m^{1/2} M_m, \quad (2.12)$$

$$D_m = \left(\sum_{|\alpha|=m} \|\rho^{-1/2} \theta_\mu \partial_h^\alpha \partial_z \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^m M_m, \quad (2.13)$$

$$Z_m = \left(\sum_{|\alpha|=m} \|Z \theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^{m-1/2} m^{1/2} M_m, \quad Z = \frac{z}{t + \mu}, \quad (2.14)$$

这里 $\tau := \tau(t)$ 是关于时间的正函数, M_m 表示为

$$M_m = \frac{(m+1)^r}{m!}, \quad r > 3. \quad (2.15)$$

特别地, 为了控制增长项 $z \mathcal{U} \partial_z \mathbf{u}$, 引入权函数 θ_μ 定义如下:

$$\theta_\mu(t, z) = \exp\left(\frac{z^2}{1 + t/\mu}\right), \quad (2.16)$$

其中 μ 为适当小的正常数. 关于权函数 θ_μ 的引入, 是本文的主要创新之处.

下面陈述本文的主要结果.

定理 2.1 设外流和初值满足以下条件.

(i) 设密度函数 $\rho(t, x, y)$ 有正的下界; $(\rho, \mathbf{U})$ 对于 $t \in [0, T]$ 是一致实解析的, 即存在正常数 C_* 以及解析半径 $\tau_* > 0$ 使得

$$\sum_{|\alpha|=m} \|\partial_h^\alpha \rho\|_{L^\infty([0, T]; L_h^\infty)} + \sum_{|\alpha|=m} \|\partial_h^\alpha \mathbf{U}\|_{L^\infty([0, T]; L_h^\infty)} \leq \frac{C_*}{\tau_*^m M_m}. \quad (2.17)$$

(ii) 给定实数 μ, λ 和 r 满足 $\mu \leq 1/C$, $\lambda > 1$ 和 $r > 3$, 其中 $C = C(C_*, r, \tau_*)$ 为适当大的正常数. 存在解析半径 $\tau_0 > 0$, 使得 $\mathbf{u}_0 \in X_{\tau_0, \mu}$.

则存在时间 $T_* > 0$, 使得初边值问题 (2.3)–(2.8) 在时间区间 $[0, T_*]$ 上存在唯一解 $\mathbf{u} \in X_{\tau, \mu}$, 其解析半径 $\tau(t) \geq \tau_0/3$.

注 2.1 定理 2.1 的证明主要包括先验估计与局部适定性的建立, 一旦建立了合适的先验估计, 关于切向解析解的局部存在性与唯一性的证明, 可采用与文献 [4, 15, 39] 相似的方法得到. 因此, 本文仅给出相应的先验估计.

3 先验估计

命题 3.1 在定理 2.1 的条件下, 初边值问题 (2.3)–(2.8) 的解满足如下先验估计:

$$\|\mathbf{u}\|_{X_{\tau,\mu}}^2 + \int_0^\tau \|\mathbf{u}(s)\|_{D_{\tau,\mu}}^2 ds \lesssim \|\mathbf{u}_0\|_{X_{\tau_0,\mu}}^2, \quad (3.1)$$

其中 $\tau(t) \geq \tau_0/3$.

证明 令 $|\alpha| = m \geq 0$, 将算子 ∂_h^α 作用于方程 (2.3), 并与 $\theta_\mu^2 \partial_h^\alpha \mathbf{u}$ 作 L^2 内积, 可得

$$\langle \partial_h^\alpha (\partial_t \mathbf{u} - \rho^{-1} \partial_z^2 \mathbf{u}), \theta_\mu^2 \partial_h^\alpha \mathbf{u} \rangle = \langle \partial_h^\alpha (\mathcal{F} - \mathcal{L}(\mathbf{u}) - \mathcal{N}(\mathbf{u})), \theta_\mu^2 \partial_h^\alpha \mathbf{u} \rangle. \quad (3.2)$$

首先, 处理方程 (3.2) 左边的项. 由 $\partial_t \theta_\mu = \theta_\mu \frac{-\mu z^2}{(t+\mu)^2} = -\mu Z^2 \theta_\mu$ (这是与代数权不一样的关键性质), 可得

$$\langle \partial_h^\alpha \partial_t \mathbf{u}, \theta_\mu^2 \partial_h^\alpha \mathbf{u} \rangle = \frac{1}{2} \frac{d}{dt} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 + \mu \|Z \theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2. \quad (3.3)$$

又由 $\partial_z \theta_\mu = \theta_\mu \frac{2\mu z}{t+\mu} = 2\mu Z \theta_\mu$ (这是与代数权不一样的另一个关键性质), 以及边界条件 (2.8), 有

$$\begin{aligned} -\langle \partial_h^\alpha (\rho^{-1} \partial_z^2 \mathbf{u}), \theta_\mu^2 \partial_h^\alpha \mathbf{u} \rangle &= -\langle \rho^{-1} \partial_h^\alpha \partial_z^2 \mathbf{u}, \theta_\mu^2 \partial_h^\alpha \mathbf{u} \rangle - \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \langle \partial_h^\beta \rho^{-1} \partial_h^{\alpha-\beta} \partial_z^2 \mathbf{u}, \theta_\mu^2 \partial_h^\alpha \mathbf{u} \rangle \\ &= \|\rho^{-1/2} \theta_\mu \partial_h^\alpha \partial_z \mathbf{u}\|_{L^2}^2 + 4\mu \langle \rho^{-1} \theta_\mu \partial_h^\alpha \partial_z \mathbf{u}, Z \theta_\mu \partial_h^\alpha \mathbf{u} \rangle \\ &\quad + \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \langle \theta_\mu \partial_h^\beta \rho^{-1} \partial_h^{\alpha-\beta} \partial_z \mathbf{u}, \theta_\mu \partial_h^\alpha \partial_z \mathbf{u} \rangle \\ &\quad + 4\mu \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \langle \theta_\mu \partial_h^\beta \rho^{-1} \partial_h^{\alpha-\beta} \partial_z \mathbf{u}, Z \theta_\mu \partial_h^\alpha \mathbf{u} \rangle. \end{aligned} \quad (3.4)$$

再将 (3.3) 和 (3.4) 代入 (3.2) 的左边, 利用 Hölder 不等式, 可得

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 + \mu \|Z \theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 + \|\rho^{-1/2} \theta_\mu \partial_h^\alpha \partial_z \mathbf{u}\|_{L^2}^2 \\ &\leq |\langle \partial_h^\alpha (\mathcal{F} - \mathcal{L}(\mathbf{u}) - \mathcal{N}(\mathbf{u})), \theta_\mu^2 \partial_h^\alpha \mathbf{u} \rangle| - 4C_* \mu \|\rho^{-1/2} \theta_\mu \partial_h^\alpha \partial_z \mathbf{u}\|_{L^2} \|Z \theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \\ &\quad + C_* \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \|\partial_h^\beta \rho^{-1}\|_{L_h^\infty} \|\rho^{-1/2} \theta_\mu \partial_h^{\alpha-\beta} \partial_z \mathbf{u}\|_{L^2} \|\rho^{-1/2} \theta_\mu \partial_h^\alpha \partial_z \mathbf{u}\|_{L^2} \\ &\quad + 4C_* \mu \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \|\partial_h^\beta \rho^{-1}\|_{L_h^\infty} \|\rho^{-1/2} \theta_\mu \partial_h^{\alpha-\beta} \partial_z \mathbf{u}\|_{L^2} \|Z \theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}. \end{aligned} \quad (3.5)$$

用 $\tau^{2m} M_m^2$ 与 (3.5) 相乘, 并对满足 $|\alpha| = m$ 的指标 α 求和, 结合 M_m , Z_m 和 D_m 的定义, 可得

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right) \tau^{2m} M_m^2 + \mu Z_m^2 + D_m^2 \\ &\leq \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha (\mathcal{F} - \mathcal{L}(\mathbf{u}) - \mathcal{N}(\mathbf{u})), \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 + 4C_* \mu Z_m D_m \\ &\quad + C_* \sum_{j=1}^m \binom{m}{j} \left(\sum_{|\beta|=j} \|\partial_h^\beta \rho^{-1}\|_{L_h^\infty} \right) \left(\sum_{|\gamma|=m-j} \|\rho^{-1/2} \theta_\mu \partial_h^\gamma \partial_z \mathbf{u}\|_{L^2}^2 \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
 & \times \left(\sum_{|\alpha|=m} \|\rho^{-1/2} \theta_\mu \partial_h^\alpha \partial_z \mathbf{u}\|_{L^2}^2 \right)^{\frac{1}{2}} \tau^{2m} M_m^2 \\
 & + 4C_* \mu \sum_{j=1}^m \binom{m}{j} \left(\sum_{|\beta|=j} \|\partial_h^\beta \rho^{-1}\|_{L_h^\infty} \right) \left(\sum_{|\gamma|=m-j} \|\rho^{-1/2} \theta_\mu \partial_h^\gamma \partial_z \mathbf{u}\|_{L^2}^2 \right)^{1/2} \\
 & \times \left(\sum_{|\alpha|=m} \|Z \theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^{2m} M_m^2.
 \end{aligned} \tag{3.6}$$

注意到, 在外流实解析的假设下, 可以验证 ρ^{-1} 和 $\mathcal{U}$ 等外流项仍为实解析函数, 且通过适当增大常数 C_* 以及减小解析半径 τ_* , 可使其均满足不等式 (2.17), 结合 Cauchy 不等式, 则有

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} (X_m^2) - \tau' Y_m^2 + (\mu - 8C_*^2 \mu^2) Z_m^2 + \frac{1}{2} D_m^2 \\
 & \leq \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha (\mathcal{F} - \mathcal{L}(\mathbf{u}) - \mathcal{N}(\mathbf{u})), \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \\
 & \quad + C_*^2 \sum_{j=1}^m \left(\frac{\tau}{\tau_*} \right)^j a_{m,r,j} D_{m-j} (D_m + 4\mu Z_m),
 \end{aligned} \tag{3.7}$$

其中

$$a_{m,r,j} = \frac{m!(m+1)^r j!(m-j)!}{j!(m-j)!(j+1)^r (m-j+1)^r} = \frac{(m+1)^r}{(j+1)^r (m-j+1)^r} \leq 1. \tag{3.8}$$

将 (3.7) 对所有的 $m \geq 0$ 求和, 利用离散的 Young 不等式

$$\|\zeta \cdot (\xi * \eta)\|_{l^1} \leq \|\zeta\|_{l^2} \|\xi\|_{l^1} \|\eta\|_{l^2}, \tag{3.9}$$

结合 (2.10), 可得

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|_{X_{\tau,\mu}}^2 - \tau' \|\mathbf{u}\|_{Y_{\tau,\mu}}^2 + (\mu - 8C_*^2 \mu^2) \|\mathbf{u}\|_{Z_{\tau,\mu}}^2 + \frac{1}{2} \|\mathbf{u}\|_{D_{\tau,\mu}}^2 \\
 & \leq \sum_{m \geq 0} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha (\mathcal{F} - \mathcal{L}(\mathbf{u}) - \mathcal{N}(\mathbf{u})), \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \\
 & \quad + C_*^2 \sum_{m \geq 1} \sum_{j=1}^m \left(\frac{\tau}{\tau_*} \right)^j D_{m-j} (D_m + 4\mu Z_m) \\
 & \leq \sum_{m \geq 0} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha (\mathcal{F} - \mathcal{L}(\mathbf{u}) - \mathcal{N}(\mathbf{u})), \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \\
 & \quad + C_*^2 \sum_{j \geq 1} \left(\frac{\tau}{\tau_*} \right)^j (\|\mathbf{u}\|_{D_{\tau,\mu}}^2 + 4\mu \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{Z_{\tau,\mu}}).
 \end{aligned} \tag{3.10}$$

选择 $\tau_0 \leq \min\{\tau_*/2, \tau_*/(1+16C_*^2)\}$, 则在 $\tau(t)$ 单调递减的假设下, 有

$$C_*^2 \sum_{j \geq 1} \left(\frac{\tau}{\tau_*} \right)^j \leq C_*^2 \sum_{j \geq 1} \left(\frac{\tau_0}{\tau_*} \right)^j = C_*^2 \frac{\tau_0}{\tau_* - \tau_0} \leq \frac{1}{16}. \tag{3.11}$$

综上, 联合 (3.10) 和 (3.11), 利用 Cauchy 不等式, 可得

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|_{X_{\tau,\mu}}^2 - \tau' \|\mathbf{u}\|_{Y_{\tau,\mu}}^2 + (\mu - C\mu^2) \|\mathbf{u}\|_{Z_{\tau,\mu}}^2 + \frac{3}{8} \|\mathbf{u}\|_{D_{\tau,\mu}}^2$$

$$\leq \sum_{m \geq 0} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha (\mathcal{F} - \mathcal{L}(\mathbf{u}) - \mathcal{N}(\mathbf{u})), \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2, \quad (3.12)$$

其中常数 $C = 8C_*^2 + 1/4$. 下面将 (3.12) 的右端分为三个部分进行估计.

首先, 给出 $\mathbf{g}$ 和 $\mathbf{u}$ 的某些估计, 其在后续证明中将多次使用. 后面, 用 $A \lesssim B$ ($A \gtrsim B$) 表示存在一个常数 C , 使得 $A \leq CB$ ($A \geq CB$).

引理 3.1 对所有的 $m \geq 0$, $|\alpha| = m$ 和 $\mu > 0$, 有如下估计:

$$\|\partial_h^\alpha \mathbf{g}\|_{L_h^2 L_z^\infty} \lesssim \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}, \quad (3.13)$$

$$\|\partial_h^\alpha \mathbf{g}\|_{L_h^\infty L_z^\infty} \lesssim \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^{1/2} \|\theta_\mu \partial_h^{\alpha+e_2} \mathbf{u}\|_{L^2}^{1/2} + \|\theta_\mu \partial_h^{\alpha+e_1} \mathbf{u}\|_{L^2}^{1/2} \|\theta_\mu \partial_h^{\alpha+e} \mathbf{u}\|_{L^2}^{1/2}, \quad (3.14)$$

$$\|\partial_h^\alpha \mathbf{u}\|_{L_h^2 L_z^\infty} \lesssim \|\theta_\mu \partial_h^\alpha \partial_z \mathbf{u}\|_{L^2}, \quad (3.15)$$

$$\|\partial_h^\alpha \mathbf{u}\|_{L_h^\infty L_z^\infty} \lesssim \|\theta_\mu \partial_h^\alpha \partial_z \mathbf{u}\|_{L^2}^{1/2} \|\theta_\mu \partial_h^{\alpha+e_2} \partial_z \mathbf{u}\|_{L^2}^{1/2} + \|\theta_\mu \partial_h^{\alpha+e_1} \partial_z \mathbf{u}\|_{L^2}^{1/2} \|\theta_\mu \partial_h^{\alpha+e} \partial_z \mathbf{u}\|_{L^2}^{1/2}, \quad (3.16)$$

其中 $e_1 = (1, 0)$, $e_2 = (0, 1)$, $e = (1, 1)$.

证明 根据 $\mathbf{g}$ 的定义, 有

$$|\partial_h^\alpha \mathbf{g}| \leq \int_0^z \theta_\mu^{-1} |\theta_\mu \partial_h^\alpha \mathbf{u}(t, x, y, \tilde{z})| d\tilde{z} \leq \int_0^{+\infty} \theta_\mu^{-1} |\theta_\mu \partial_h^\alpha \mathbf{u}(t, x, y, \tilde{z})| d\tilde{z}. \quad (3.17)$$

又当 $\mu > 0$ 时, $\theta_\mu^{-1} \in L_z^2$, 可得

$$\|\partial_h^\alpha \mathbf{g}\|_{L_z^\infty} \leq \|\theta_\mu^{-1}\|_{L_z^2} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L_z^2} \lesssim \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L_z^2}. \quad (3.18)$$

对 (3.18) 两边关于切向方向取 L^2 范数, 即得 (3.13).

而对 (3.18) 两边关于切向方向取 L^∞ 范数, 再利用如下的二维 Agmon 不等式:

$$\|f\|_{L_{x,y}^\infty} \lesssim \|f\|_{L_{x,y}^2}^{1/2} \|\partial_y f\|_{L_{x,y}^2}^{1/2} + \|\partial_x f\|_{L_{x,y}^2}^{1/2} \|\partial_x \partial_y f\|_{L_{x,y}^2}^{1/2}, \quad (3.19)$$

即可得到 (3.14).

由于 $\mathbf{u} = \int_0^z \partial_z \mathbf{u} d\tilde{z}$, 利用类似的方法可直接得到 (3.15) 和 (3.16). 引理 3.1 证毕. $\square$

3.1 线性项的估计

引理 3.2 给定实数 $\mu > 0$ 和 $\lambda > 1$, 有如下估计:

$$\begin{aligned} L &= \sum_{m \geq 0} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha \mathcal{L}(\mathbf{u}), \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \\ &\leq \frac{1}{8} \|\mathbf{u}\|_{D_{\tau,\mu}}^2 + C \left(\frac{1}{\tau} \|\mathbf{u}\|_{X_{\tau,\mu}}^2 + \|\mathbf{u}\|_{X_{\tau,\mu}}^2 + \|\mathbf{u}\|_{Y_{\tau,\mu}}^2 + \mu^2 \|\mathbf{u}\|_{Z_{\tau,\mu}}^2 \right), \end{aligned} \quad (3.20)$$

其中 $C = C(C_*, r, \tau_*)$ 为适当的正常数.

证明 由 (2.4), 记

$$\begin{aligned} \mathcal{L}(\mathbf{u}) &= -z\mathcal{U}\partial_z \mathbf{u} + (1-\phi)\mathbf{u} \cdot \nabla \mathbf{U} + (1-\phi)\mathbf{U} \cdot \nabla \mathbf{u} + \phi'(\nabla \tilde{\rho} \cdot \mathbf{g})\mathbf{U} \\ &\quad + \phi'(\nabla \cdot \mathbf{g})\mathbf{U} + \Phi(\nabla \tilde{\rho} \cdot \mathbf{U} + \nabla \cdot \mathbf{U})\partial_z \mathbf{u} \\ &=: \sum_{i=1}^6 \mathcal{L}_i. \end{aligned} \quad (3.21)$$

令

$$L_i = \sum_{m \geq 0} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha \mathcal{L}_i, \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2, \quad (3.22)$$

则有

$$L = \sum_{m \geq 0} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha \mathcal{L}(\mathbf{u}), \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \leq \sum_{i=1}^6 L_i. \quad (3.23)$$

接下来对 (3.23) 右端进行逐项估计. 首先, 对于 L_1 , 有

$$\begin{aligned} L_1 &= \sum_{m \geq 0} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha (z \mathcal{U} \partial_z \mathbf{u}), \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \\ &\leq C_*(t + \mu) \sum_{m \geq 0} \sum_{|\alpha|=m} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \|\partial_h^\beta \mathcal{U}\|_{L_h^\infty} \|\rho^{-1/2} \theta_\mu \partial_h^{\alpha-\beta} \partial_z \mathbf{u}\|_{L^2} \|Z \theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \\ &\leq C_*(t + \mu) \sum_{m \geq 0} \sum_{j=0}^m \binom{m}{j} \left(\sum_{|\beta|=j} \|\partial_h^\beta \mathcal{U}\|_{L_h^\infty} \right) \left(\sum_{|\gamma|=m-j} \|\rho^{-1/2} \theta_\mu \partial_h^\gamma \partial_z \mathbf{u}\|_{L^2}^2 \right)^{1/2} \\ &\quad \times \left(\sum_{|\alpha|=m} \|Z \theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^{2m} M_m^2 \\ &\leq C_*^2(t + \mu) \sum_{m \geq 0} \sum_{j=0}^m \left(\frac{\tau}{\tau_*} \right)^j a_{m,r,j} D_{m-j} Z_m. \end{aligned} \quad (3.24)$$

假设 T 足够小使得 $T \leq \mu$ 成立, 结合估计 (2.10), (3.8) 和 (3.11), 并利用离散的 Young 不等式, 可得

$$L_1 \leq \frac{1}{16} \|\mathbf{u}\|_{D_{\tau,\mu}}^2 + C\mu^2 \|\mathbf{u}\|_{Z_{\tau,\mu}}^2, \quad (3.25)$$

其中 C 为适当大的正常数.

下面给出 L_2 的估计. 根据 ϕ 的定义可知, 当 $\mu > 0$ 时有 $\phi \in L_z^\infty$, 可得

$$\begin{aligned} L_2 &= \sum_{m \geq 0} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha ((1 - \phi) \mathbf{u} \cdot \nabla \mathbf{U}), \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \\ &\leq \sum_{m \geq 0} \sum_{|\alpha|=m} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \|1 - \phi\|_{L_z^\infty} \|\partial_h^\beta \nabla \mathbf{U}\|_{L_h^\infty} \|\theta_\mu \partial_h^{\alpha-\beta} \mathbf{u}\|_{L^2} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2 \\ &\leq \sum_{m \geq 0} \sum_{j=0}^m \binom{m}{j} \left(\sum_{|\beta|=j} \|\partial_h^\beta \nabla \mathbf{U}\|_{L_h^\infty} \right) \left(\sum_{|\gamma|=m-j} \|\theta_\mu \partial_h^\gamma \mathbf{u}\|_{L^2}^2 \right)^{1/2} \\ &\quad \times \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^{2m} M_m^2 \\ &\leq C_* \sum_{m \geq 0} \sum_{j=0}^m \left(\frac{\tau}{\tau_*} \right)^j a_{m,r,j} X_{m-j} X_m \\ &\lesssim \|\mathbf{u}\|_{X_{\tau,\mu}}^2. \end{aligned} \quad (3.26)$$

L_3 可分解成如下两部分:

$$\begin{aligned} L_3 &= |\langle \theta_\mu(1-\phi)(\mathbf{U} \cdot \nabla \mathbf{u}), \theta_\mu \mathbf{u} \rangle| + \sum_{m \geq 1} \sum_{|\alpha|=m} |\langle \theta_\mu(1-\psi) \partial_h^\alpha (\mathbf{U} \cdot \nabla \mathbf{u}), \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \\ &=: L_3^1 + L_3^2. \end{aligned} \quad (3.27)$$

根据 Hölder 不等式, 可得

$$L_3^1 \leq \|1-\phi\|_{L_z^\infty} \|\mathbf{U}\|_{L_h^\infty} \|\theta_\mu \nabla \mathbf{u}\|_{L^2} \|\theta_\mu \mathbf{u}\|_{L^2} \lesssim \frac{1}{\tau} X_1 X_0 \lesssim \frac{1}{\tau} \|\mathbf{u}\|_{X_{\tau,\mu}}^2, \quad (3.28)$$

以及

$$\begin{aligned} L_3^2 &\leq \sum_{m \geq 1} \sum_{|\alpha|=m} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \|1-\phi\|_{L_z^\infty} \|\partial_h^\beta \mathbf{U}\|_{L_h^\infty} \|\theta_\mu \partial_h^{\alpha-\beta} \nabla \mathbf{u}\|_{L^2} \\ &\quad \times \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2 \\ &\leq \sum_{m \geq 1} \sum_{j=0}^m \binom{m}{j} \left(\sum_{|\beta|=j} \|\partial_h^\beta \mathbf{U}\|_{L_h^\infty} \right) \left(\sum_{|\gamma|=m-j+1} \|\theta_\mu \partial_h^\gamma \mathbf{u}\|_{L^2}^2 \right)^{1/2} \\ &\quad \times \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^{2m} M_m^2 \\ &\leq C_* \sum_{m \geq 1} \sum_{j=0}^m \left(\frac{\tau}{\tau_*} \right)^j b_{m,r,j} Y_{m-j+1} Y_m. \end{aligned} \quad (3.29)$$

由于对所有的 $0 \leq j \leq m$ 和 $m \geq 1$, 有

$$\begin{aligned} b_{m,r,j} &= \frac{\binom{m}{j} M_m}{M_j M_{m-j+1} (m-j+1)^{1/2} m^{1/2}} \\ &\leq \frac{(m+1)^r}{(j+1)^r (m-j+1)^r} \frac{(m-j+1)^{1/2}}{m^{1/2}} \\ &\leq C, \end{aligned} \quad (3.30)$$

再利用离散的 Young 不等式, 可得

$$L_3^2 \lesssim \|\mathbf{u}\|_{Y_{\tau,\mu}}^2. \quad (3.31)$$

综合 (3.28) 和 (3.31), 有

$$L_3 \lesssim \frac{1}{\tau} \|\mathbf{u}\|_{X_{\tau,\mu}}^2 + \|\mathbf{u}\|_{Y_{\tau,\mu}}^2. \quad (3.32)$$

接下来, 给出 L_4 的估计:

$$\begin{aligned} L_4 &= \sum_{m \geq 0} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha (\phi'(\nabla \tilde{\rho} \cdot \mathbf{g}) \mathbf{U}), \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \\ &\leq \sum_{m \geq 0} \sum_{|\alpha|=m} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \|\theta_\mu \phi'\|_{L_z^2} \|\partial_h^\beta (\nabla \tilde{\rho} \otimes \mathbf{U})\|_{L_h^\infty} \|\theta_\mu \partial_h^{\alpha-\beta} \mathbf{g}\|_{L_h^2 L_z^\infty} \\ &\quad \times \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2. \end{aligned} \quad (3.33)$$

注意到

$$\phi' = \frac{-2\sqrt{\lambda}}{\sqrt{\pi}} \exp(-\lambda z^2), \quad \theta_\mu \phi' = \frac{-2\sqrt{\lambda}}{\sqrt{\pi}} \exp\left[\left(-\lambda + \frac{\mu}{t+\mu}\right)z^2\right], \quad (3.34)$$

则当 $\lambda > 1$ 时, 有 $\theta_\mu \phi' \in L_z^2$. 结合引理 3.1, 可得

$$\begin{aligned} L_4 &\lesssim \sum_{m \geq 0} \sum_{j=0}^m \binom{m}{j} \left(\sum_{|\beta|=j} \|\partial_h^\beta (\nabla \tilde{\rho} \otimes U)\|_{L_h^\infty} \right) \left(\sum_{|\gamma|=m-j} \|\theta_\mu \partial_h^\gamma \mathbf{u}\|_{L^2}^2 \right)^{1/2} \\ &\quad \times \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^{2m} M_m^2 \\ &\lesssim \sum_{m \geq 0} \sum_{j=0}^m \left(\frac{\tau}{\tau_*} \right)^j a_{m,r,j} X_{m-j} X_m \\ &\lesssim \|\mathbf{u}\|_{X_{\tau,\mu}}^2. \end{aligned} \quad (3.35)$$

对于 L_5 , 可作如下分解:

$$\begin{aligned} L_5 &= |\langle \theta_\mu \phi' (\nabla \cdot \mathbf{g}) U, \theta_\mu \mathbf{u} \rangle| + \sum_{m \geq 1} \sum_{|\alpha|=m} |\langle \theta_\mu \psi' \partial_h^\alpha [(\nabla \cdot \mathbf{g}) U], \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \\ &=: L_5^1 + L_5^2. \end{aligned} \quad (3.36)$$

利用引理 3.1 和 Hölder 不等式, 可得

$$\begin{aligned} L_5^1 &\leq \|\theta_\mu \phi'\|_{L_z^2} \|\nabla \cdot \mathbf{g}\|_{L_h^2 L_z^\infty} \|U\|_{L_h^\infty} \|\theta_\mu \mathbf{u}\|_{L^2} \\ &\lesssim \|\theta_\mu \nabla \mathbf{u}\|_{L^2} \|\theta_\mu \mathbf{u}\|_{L^2} \lesssim \frac{1}{\tau} X_1 X_0 \lesssim \frac{1}{\tau} \|\mathbf{u}\|_{X_{\tau,\mu}}^2, \end{aligned} \quad (3.37)$$

以及

$$\begin{aligned} L_5^2 &\leq \sum_{m \geq 1} \sum_{|\alpha|=m} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \|\theta_\mu \phi'\|_{L_z^2} \|\partial_h^\beta U\|_{L_h^\infty} \|\theta_\mu \partial_h^{\alpha-\beta} (\nabla \cdot \mathbf{g})\|_{L_h^2 L_z^\infty} \\ &\quad \times \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2 \\ &\lesssim \sum_{m \geq 1} \sum_{j=0}^m \binom{m}{j} \left(\sum_{|\beta|=j} \|\partial_h^\beta U\|_{L_h^\infty} \right) \left(\sum_{|\gamma|=m-j+1} \|\theta_\mu \partial_h^\gamma \mathbf{u}\|_{L^2}^2 \right)^{1/2} \\ &\quad \times \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^{2m} M_m^2 \\ &\lesssim \sum_{m \geq 1} \sum_{j=0}^m \left(\frac{\tau}{\tau_*} \right)^j b_{m,r,j} Y_{m-j+1} Y_m \lesssim \|\mathbf{u}\|_{Y_{\tau,\mu}}^2. \end{aligned} \quad (3.38)$$

联合 (3.37) 和 (3.38), 可得

$$L_5 \lesssim \frac{1}{\tau} \|\mathbf{u}\|_{X_{\tau,\mu}}^2 + \|\mathbf{u}\|_{Y_{\tau,\mu}}^2. \quad (3.39)$$

最后, 给出 L_6 的估计:

$$\begin{aligned} L_6 &\lesssim \sum_{m \geq 0} \sum_{|\alpha|=m} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \|\Phi\|_{L_z^\infty} (\|\partial_h^\beta (\nabla \tilde{\rho} \cdot \mathbf{U})\|_{L_h^\infty} + \|\partial_h^\beta (\nabla \cdot \mathbf{U})\|_{L_h^\infty}) \\ &\quad \times \|\rho^{-1/2} \theta_\mu \partial_h^{\alpha-\beta} \partial_z \mathbf{u}\|_{L^2} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2. \end{aligned} \quad (3.40)$$

又根据 Φ 的定义, 当 $\lambda > 0$ 时, $\Phi \in L_z^\infty$, 可得

$$\begin{aligned} L_6 &\lesssim \sum_{m \geq 0} \sum_{j=0}^m \binom{m}{j} \left(\sum_{|\beta|=j} \|\partial_h^\beta (\nabla \tilde{\rho} \cdot \mathbf{U})\|_{L_h^\infty} + \sum_{|\beta|=j} \|\partial_h^\beta (\nabla \cdot \mathbf{U})\|_{L_h^\infty} \right) \\ &\quad \times \left(\sum_{|\gamma|=m-j} \|\rho^{-1/2} \theta_\mu \partial_h^\gamma \partial_z \mathbf{u}\|_{L^2}^2 \right)^{1/2} \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^{2m} M_m^2 \\ &\lesssim \sum_{m \geq 0} \sum_{j=0}^m \left(\frac{\tau}{\tau_*} \right)^j a_{m,r,j} D_{m-j} X_m \\ &\lesssim \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{X_{\tau,\mu}} \leq \frac{1}{16} \|\mathbf{u}\|_{D_{\tau,\mu}}^2 + C \|\mathbf{u}\|_{X_{\tau,\mu}}^2. \end{aligned} \quad (3.41)$$

综合 L_1 – L_6 的估计, 引理 3.2 得证. $\square$

3.2 非线性项的估计

引理 3.3 给定实数 $\mu > 0$ 和 $r > 3$, 有如下估计:

$$\begin{aligned} N &= \sum_{m \geq 0} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha \mathcal{N}, \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \\ &\leq \frac{1}{8} \|\mathbf{u}\|_{D_{\tau,\mu}}^2 + \frac{C}{\tau^4} \|\mathbf{u}\|_{X_{\tau,\mu}}^4 + \frac{C}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{Y_{\tau,\mu}}^2, \end{aligned}$$

其中 $C = C(C_*, r, \tau_*)$ 为适当的正常数.

证明 由 (2.5), 记

$$\mathcal{N}(\mathbf{u}) = -(\nabla \cdot \mathbf{g}) \partial_z \mathbf{u} - (\nabla \tilde{\rho} \cdot \mathbf{g}) \partial_z \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} =: \sum_{i=1}^3 \mathcal{N}_i. \quad (3.42)$$

令

$$N_i = \sum_{m \geq 0} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha \mathcal{N}_i, \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2, \quad (3.43)$$

则有

$$N = \sum_{m \geq 0} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha \mathcal{N}, \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \leq \sum_{i=1}^3 N_i. \quad (3.44)$$

接下来, 对 N_1 – N_3 进行逐项估计.

对于 N_1 , 作如下分解:

$$N_1 = |\langle \theta_\mu (\nabla \cdot \mathbf{g}) \partial_z \mathbf{u}, \theta_\mu \mathbf{u} \rangle| + \sum_{m \geq 1} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha [(\nabla \cdot \mathbf{g}) \partial_z \mathbf{u}], \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2$$

$$=: N_1^1 + N_1^2. \quad (3.45)$$

由引理 3.1 可得

$$\begin{aligned} N_1^1 &\leq \|\nabla \cdot \mathbf{g}\|_{L_h^2 L_z^\infty} \|\theta_\mu \partial_z \mathbf{u}\|_{L_h^\infty L_z^2} \|\theta_\mu \mathbf{u}\|_{L^2} \\ &\lesssim \|\theta_\mu \nabla \mathbf{u}\|_{L^2} (\|\rho^{-1/2} \theta_\mu \partial_z \mathbf{u}\|_{L^2}^{1/2} \|\rho^{-1/2} \theta_\mu \partial_y \partial_z \mathbf{u}\|_{L^2}^{1/2} \|\theta_\mu \mathbf{u}\|_{L^2} \\ &\quad + \|\rho^{-1/2} \theta_\mu \partial_x \partial_z \mathbf{u}\|_{L^2}^{1/2} \|\rho^{-1/2} \theta_\mu \partial_x \partial_y \partial_z \mathbf{u}\|_{L^2}^{1/2} \|\theta_\mu \mathbf{u}\|_{L^2}) \\ &\lesssim \frac{1}{\tau} Y_1 D_0^{1/2} D_1^{1/2} X_0 + \frac{1}{\tau^2} Y_1 D_1^{1/2} D_2^{1/2} X_0 \\ &\lesssim \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{Y_{\tau,\mu}} \|\mathbf{u}\|_{X_{\tau,\mu}}. \end{aligned} \quad (3.46)$$

需要强调, (3.46) 用到了 $\tau \leq \tau_*/2$.

对于 N_1^2 , 有

$$\begin{aligned} N_1^2 &= \sum_{m \geq 1} \sum_{|\alpha|=m} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} |\langle \theta_\mu \partial_h^{\alpha-\beta} (\nabla \cdot \mathbf{g}) \partial_h^\beta \partial_z \mathbf{u}, \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \\ &\leq N_{1,1}^2 + N_{1,2}^2, \end{aligned} \quad (3.47)$$

其中

$$\begin{aligned} N_{1,1}^{2,1} &= \sum_{m \geq 1} \sum_{j=0}^{[m/2]} \sum_{|\alpha|=m} \sum_{|\beta|=j, \beta \leq \alpha} \binom{m}{j} \|\partial_h^{\alpha-\beta} (\nabla \cdot \mathbf{g})\|_{L_h^2 L_z^\infty} \|\theta_\mu \partial_h^\beta \partial_z \mathbf{u}\|_{L_h^\infty L_z^2} \\ &\quad \times \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2, \end{aligned} \quad (3.48)$$

$$\begin{aligned} N_{1,1}^{2,2} &= \sum_{m \geq 1} \sum_{j=[m/2]+1}^m \sum_{|\alpha|=m} \sum_{|\beta|=j, \beta \leq \alpha} \binom{m}{j} \|\partial_h^{\alpha-\beta} (\nabla \cdot \mathbf{g})\|_{L^\infty} \|\theta_\mu \partial_h^\beta \partial_z \mathbf{u}\|_{L^2} \\ &\quad \times \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2. \end{aligned} \quad (3.49)$$

借助引理 3.1, 可得

$$\begin{aligned} N_{1,1}^{2,1} &\lesssim \sum_{m \geq 1} \sum_{j=0}^{[m/2]} \binom{m}{j} \sum_{|\alpha|=m} \sum_{|\beta|=j, \beta \leq \alpha} \|\theta_\mu \partial_h^{\alpha-\beta} \nabla \mathbf{u}\|_{L^2} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \\ &\quad \times (\|\rho^{-1/2} \theta_\mu \partial_h^\beta \partial_z \mathbf{u}\|_{L^2}^{1/2} \|\rho^{-1/2} \theta_\mu \partial_h^{\beta+e_2} \partial_z \mathbf{u}\|_{L^2}^{1/2} \\ &\quad + \|\rho^{-1/2} \theta_\mu \partial_h^{\beta+e_1} \partial_z \mathbf{u}\|_{L^2}^{1/2} \|\rho^{-1/2} \theta_\mu \partial_h^{\beta+e} \partial_z \mathbf{u}\|_{L^2}^{1/2}) \tau^{2m} M_m^2 \\ &\lesssim \sum_{m \geq 1} \sum_{j=0}^{[m/2]} \binom{m}{j} \left(\sum_{|\gamma|=m-j+1} \|\theta_\mu \partial_h^\gamma \mathbf{u}\|_{L^2}^2 \right)^{1/2} \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \\ &\quad \times \sum_{|\beta|=j}^{j+2} \|\rho^{-1/2} \theta_\mu \partial_h^\beta \partial_z \mathbf{u}\|_{L^2} \tau^{2m} M_m^2. \end{aligned} \quad (3.50)$$

注意到

$$\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \leq (m+1)^{1/2} \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2}, \quad (3.51)$$

则有

$$N_1^{2,1} \lesssim \sum_{m \geq 1} \sum_{j=0}^{[m/2]} \sum_{i=1}^3 \frac{c_{m,r,j,i} Y_{m-j+1} D_{j+i-1} Y_m}{\tau^{i-1}}. \quad (3.52)$$

对于所有的 $0 \leq j \leq [m/2]$ 和 $1 \leq i \leq 3$, 有

$$\begin{aligned} c_{m,r,j,i} &= \frac{\binom{m}{j} (j+i)^{1/2} M_m}{(m-j+1)^{1/2} M_{m-j+1} M_{j+i-1} m^{1/2}} \\ &\leq \frac{(m+1)^r}{(m-j+2)^r} \frac{(m-j+1)^{1/2}}{m^{1/2}} (j+i)^{i-1/2-r} \\ &\lesssim (j+i)^{i-1/2-r}, \end{aligned} \quad (3.53)$$

因此, 当 $r > 3$ 时, 结合离散的 Young 不等式, 可得

$$N_1^{2,1} \lesssim \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{Y_{\tau,\mu}}^2. \quad (3.54)$$

类似地, 利用引理 3.1, 有

$$\begin{aligned} N_1^{2,2} &\lesssim \sum_{m \geq 1} \sum_{j=[m/2]+1}^m \binom{m}{j} \sum_{|\alpha|=m} \sum_{|\beta|=j, \beta \leq \alpha} \|\rho^{-1/2} \theta_\mu \partial_h^\beta \partial_z \mathbf{u}\|_{L^2} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2 \\ &\quad \times (\|\theta_\mu \partial_h^{\alpha-\beta} \nabla \mathbf{u}\|_{L^2}^{1/2} \|\theta_\mu \partial_h^{\alpha-\beta+e_2} \nabla \mathbf{u}\|_{L^2}^{1/2} \\ &\quad + \|\theta_\mu \partial_h^{\alpha-\beta+e_1} \nabla \mathbf{u}\|_{L^2}^{1/2} \|\theta_\mu \partial_h^{\alpha-\beta+e} \nabla \mathbf{u}\|_{L^2}^{1/2}) \tau^{2m} M_m^2 \\ &\lesssim \sum_{m \geq 1} \sum_{j=[m/2]+1}^m \binom{m}{j} \left(\sum_{|\beta|=j} \|\rho^{-1/2} \theta_\mu \partial_h^\beta \partial_z \mathbf{u}\|_{L^2}^2 \right)^{1/2} \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \\ &\quad \times \sum_{|\gamma|=m-j}^{m-j+2} \|\theta_\mu \partial_h^\gamma \nabla \mathbf{u}\|_{L^2} \tau^{2m} M_m^2 \\ &\lesssim \sum_{m \geq 1} \sum_{j=[m/2]+1}^m \sum_{i=1}^3 \frac{d_{m,r,j,i} D_j Y_{m-j+i} Y_m}{\tau^{i-1}}. \end{aligned} \quad (3.55)$$

对于所有的 $[m/2] + 1 \leq j \leq m$ 和 $1 \leq i \leq 3$, 有

$$\begin{aligned} d_{m,r,j,i} &= \frac{\binom{m}{j} M_m}{M_{m-j+i} M_j m^{1/2}} \leq \frac{(m+1)^r}{(j+1)^r} \frac{(m-j+i)^{1/2}}{m^{1/2}} \frac{(m-j+i)^{i-1/2}}{(m-j+i+1)^r} \\ &\lesssim (m-j+i)^{i-1/2-r}, \end{aligned} \quad (3.56)$$

则当 $r > 3$ 时, 利用离散的 Young 不等式, 可得

$$N_1^{2,2} \lesssim \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{Y_{\tau,\mu}}^2. \quad (3.57)$$

联合 (3.46), (3.54) 和 (3.57), 可得 N_1 的估计

$$N_1 \lesssim \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{Y_{\tau,\mu}}^2 + \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{Y_{\tau,\mu}} \|\mathbf{u}\|_{X_{\tau,\mu}}. \quad (3.58)$$

下面转向 N_2 的估计:

$$\begin{aligned} N_2 &= \sum_{m \geq 0} \sum_{|\alpha|=m} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} |\langle \theta_\mu \partial_h^\beta (\nabla \tilde{\rho} \cdot \mathbf{g}) \partial_h^{\alpha-\beta} \partial_z \mathbf{u}, \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \\ &\leq N_2^1 + N_2^2, \end{aligned} \quad (3.59)$$

其中

$$\begin{aligned} N_2^1 &= \sum_{m \geq 0} \sum_{j=0}^{[m/2]} \binom{m}{j} \sum_{|\alpha|=m} \sum_{|\beta|=j, \beta \leq \alpha} \left[\sum_{0 \leq \gamma \leq \beta} \binom{\beta}{\gamma} \|\partial_h^\gamma \nabla \tilde{\rho}\|_{L_h^\infty} \|\partial_h^{\beta-\gamma} \mathbf{g}\|_{L^\infty} \right] \\ &\quad \times \|\theta_\mu \partial_h^{\alpha-\beta} \partial_z \mathbf{u}\|_{L^2} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2, \end{aligned} \quad (3.60)$$

$$\begin{aligned} N_2^2 &= \sum_{m \geq 0} \sum_{j=[m/2]+1}^m \binom{m}{j} \sum_{|\alpha|=m} \sum_{|\beta|=j, \beta \leq \alpha} \left[\sum_{0 \leq \gamma \leq \beta} \binom{\beta}{\gamma} \|\partial_h^\gamma \nabla \tilde{\rho}\|_{L_h^\infty} \|\partial_h^{\beta-\gamma} \mathbf{g}\|_{L_h^2 L_z^\infty} \right] \\ &\quad \times \|\theta_\mu \partial_h^{\alpha-\beta} \partial_z \mathbf{u}\|_{L_h^\infty L_z^2} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2. \end{aligned} \quad (3.61)$$

对于 N_2^1 , 根据 Young 不等式, 有

$$\begin{aligned} N_2^1 &\lesssim \sum_{m \geq 0} \sum_{j=0}^{[m/2]} \binom{m}{j} \left(\sum_{|\beta|=j} \sum_{0 \leq \gamma \leq \beta} \binom{\beta}{\gamma} \|\partial_h^\gamma \nabla \tilde{\rho}\|_{L_h^\infty} \|\partial_h^{\beta-\gamma} \mathbf{g}\|_{L^\infty} \right) \\ &\quad \times \left(\sum_{|\nu|=m-j} \|\rho^{-1/2} \theta_\mu \partial_h^\nu \partial_z \mathbf{u}\|_{L^2}^2 \right)^{1/2} \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^{2m} M_m^2. \end{aligned} \quad (3.62)$$

由引理 3.1, 可得

$$\begin{aligned} &\sum_{|\beta|=j} \sum_{0 \leq \gamma \leq \beta} \binom{\beta}{\gamma} \|\partial_h^\gamma \nabla \tilde{\rho}\|_{L_h^\infty} \|\partial_h^{\beta-\gamma} \mathbf{g}\|_{L^\infty} \\ &\leq \sum_{i=0}^j \binom{j}{i} \sum_{|\beta|=j} \sum_{|\gamma|=i, \gamma \leq \beta} \|\partial_h^\gamma \nabla \tilde{\rho}\|_{L_h^\infty} \times (\|\theta_\mu \partial_h^{\beta-\gamma} \mathbf{u}\|_{L^2}^{1/2} \|\theta_\mu \partial_h^{\beta-\gamma+e_2} \mathbf{u}\|_{L^2}^{1/2} \\ &\quad + \|\theta_\mu \partial_h^{\beta-\gamma+e_1} \mathbf{u}\|_{L^2}^{1/2} \|\theta_\mu \partial_h^{\beta-\gamma+e} \mathbf{u}\|_{L^2}^{1/2}) \\ &\lesssim \sum_{i=0}^j \binom{j}{i} \sum_{|\gamma|=i} \|\partial_h^\gamma \nabla \tilde{\rho}\|_{L_h^\infty} \sum_{|\delta|=j-i}^{j-i+2} \|\theta_\mu \partial_h^\delta \mathbf{u}\|_{L^2}. \end{aligned} \quad (3.63)$$

将 (3.63) 代入 (3.62), 并结合 (3.51), 可得

$$N_2^1 \lesssim \sum_{m \geq 0} \sum_{j=0}^{[m/2]} \sum_{i=0}^j \left(\frac{\tau}{\tau_*} \right)^i \sum_{k=1}^3 \frac{d_{m,r,j,i,k} D_{m-j} X_m X_{j-i+k}}{\tau^{k-1}}, \quad (3.64)$$

其中

$$d_{m,r,j,i,k} = \frac{\binom{m}{j} \binom{j}{i} (j-i+k)^{1/2} M_m}{M_i M_{j-i} M_{m-j+k-1}}. \quad (3.65)$$

对于所有的 $0 \leq j \leq [m/2]$ 和 $1 \leq k \leq 3$, 直接计算可得

$$d_{m,r,j,i,k} \lesssim (j-i+k)^{k-1/2-r}. \quad (3.66)$$

因此, 当 $r > 3$ 时, 利用离散的 Young 不等式 (3.9), 有

$$N_2^1 \lesssim \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{X_{\tau,\mu}}^2. \quad (3.67)$$

对于 N_2^2 , 有

$$\begin{aligned} N_2^2 &\lesssim \sum_{m \geq 0} \sum_{j=[m/2]+1}^m \sum_{i=0}^j \binom{m}{j} \binom{j}{i} \left[\sum_{|\beta|=j} \left(\sum_{|\gamma|=i, \gamma \leq \beta} \|\partial_h^\gamma \nabla \tilde{\rho}\|_{L_h^\infty} \|\theta_\mu \partial_h^{\beta-\gamma} \mathbf{u}\|_{L^2} \right)^2 \right]^{1/2} \\ &\quad \times \sum_{|\nu|=m-j}^{m-j+2} \|\rho^{-1/2} \theta_\mu \partial_h^\nu \partial_z \mathbf{u}\|_{L^2} \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^{2m} M_m^2 \\ &\lesssim \sum_{m \geq 0} \sum_{j=[m/2]+1}^m \sum_{i=0}^j \binom{m}{j} \binom{j}{i} \left(\sum_{|\gamma|=i} \|\partial_h^\gamma \nabla \tilde{\rho}\|_{L_h^\infty} \right) \left(\sum_{|\delta|=j-i} \|\theta_\mu \partial_h^\delta \mathbf{u}\|_{L^2}^2 \right)^{1/2} \\ &\quad \times \sum_{|\nu|=m-j}^{m-j+2} \|\rho^{-1/2} \theta_\mu \partial_h^\nu \partial_z \mathbf{u}\|_{L^2} \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^{2m} M_m^2 \\ &\lesssim \sum_{m \geq 0} \sum_{j=[m/2]+1}^m \sum_{i=0}^j \left(\frac{\tau}{\tau_*} \right)^i \sum_{k=1}^3 \frac{e_{m,r,j,i,k} X_{j-i} D_{m-j+k-1} X_m}{\tau^{k-1}}. \end{aligned} \quad (3.68)$$

对所有的 $[m/2] + 1 \leq j \leq m$ 和 $1 \leq k \leq 3$, 有

$$e_{m,r,j,i,k} = \frac{\binom{m}{j} \binom{j}{i} (m-j+k)^{1/2} M_m}{M_i M_{j-i} M_{m-j+k-1}} \lesssim (m-j+k)^{k-1/2-r}. \quad (3.69)$$

联合 (3.68) 和 (3.69), 并利用 (3.9), 可得

$$N_2^2 \lesssim \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{X_{\tau,\mu}}^2. \quad (3.70)$$

综合 (3.67) 和 (3.70), 得到 N_2 的估计为

$$N_2 \lesssim \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{X_{\tau,\mu}}^2. \quad (3.71)$$

最后, 给出 N_3 的估计:

$$\begin{aligned} N_3 &= |\langle \theta_\mu (\mathbf{u} \cdot \nabla \mathbf{u}), \theta_\mu \mathbf{u} \rangle| + \sum_{m \geq 1} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha (\mathbf{u} \cdot \nabla \mathbf{u}), \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \\ &=: N_3^1 + N_3^2. \end{aligned} \quad (3.72)$$

由引理 3.1, 可得

$$\begin{aligned} N_3^1 &\leq \|\mathbf{u}\|_{L^\infty} \|\theta_\mu \nabla \mathbf{u}\|_{L^2} \|\theta_\mu \mathbf{u}\|_{L^2} \\ &\lesssim \|\rho^{-1/2} \theta_\mu \partial_z \mathbf{u}\|_{L^2}^{1/2} \|\rho^{-1/2} \theta_\mu \partial_y \partial_z \mathbf{u}\|_{L^2}^{1/2} \|\theta_\mu \nabla \mathbf{u}\|_{L^2} \|\theta_\mu \mathbf{u}\|_{L^2} \end{aligned}$$

$$\begin{aligned}
 & + \|\rho^{-1/2}\theta_\mu\partial_x\partial_z\mathbf{u}\|_{L^2}^{1/2}\|\rho^{-1/2}\theta_\mu\partial_x\partial_y\partial_z\mathbf{u}\|_{L^2}^{1/2}\|\theta_\mu\nabla\mathbf{u}\|_{L^2}\|\theta_\mu\mathbf{u}\|_{L^2} \\
 & \lesssim \frac{D_0^{1/2}D_1^{1/2}Y_1X_0}{\tau} + \frac{D_1^{1/2}D_2^{1/2}Y_1X_0}{\tau^2} \\
 & \lesssim \frac{1}{\tau^2}\|\mathbf{u}\|_{D_{\tau,\lambda}}\|\mathbf{u}\|_{Y_{\tau,\lambda}}\|\mathbf{u}\|_{X_{\tau,\lambda}}.
 \end{aligned} \tag{3.73}$$

对于 N_3^2 , 有

$$N_3^2 = \sum_{m \geq 1} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha (\mathbf{u} \cdot \nabla \mathbf{u}), \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \leq N_3^{2,1} + N_3^{2,2}, \tag{3.74}$$

其中

$$\begin{aligned}
 N_3^{2,1} &= \sum_{m \geq 1} \sum_{j=0}^{[m/2]} \sum_{|\alpha|=m} \sum_{|\beta|=j, \beta \leq \alpha} \binom{m}{j} \|\partial_h^{\alpha-\beta} \mathbf{u}\|_{L_h^2 L_z^\infty} \|\theta_\mu \partial_h^\beta \nabla \mathbf{u}\|_{L_h^\infty L_z^2} \\
 &\quad \times \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2,
 \end{aligned} \tag{3.75}$$

$$\begin{aligned}
 N_3^{2,2} &= \sum_{m \geq 1} \sum_{j=[m/2]+1}^m \sum_{|\alpha|=m} \sum_{|\beta|=j, \beta \leq \alpha} \binom{m}{j} \|\partial_h^{\alpha-\beta} \mathbf{u}\|_{L^\infty} \|\theta_\mu \partial_h^\beta \nabla \mathbf{u}\|_{L^2} \\
 &\quad \times \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2.
 \end{aligned} \tag{3.76}$$

对于 $N_3^{2,1}$, 根据引理 3.1 和 (3.51), 可得

$$\begin{aligned}
 N_3^{2,1} &\lesssim \sum_{m \geq 1} \sum_{j=0}^{[m/2]} \binom{m}{j} \sum_{|\alpha|=m} \sum_{|\beta|=j, \beta \leq \alpha} \|\rho^{-1/2}\theta_\mu\partial_h^{\alpha-\beta}\partial_z\mathbf{u}\|_{L^2} \|\theta_\mu\partial_h^\alpha\mathbf{u}\|_{L^2} \tau^{2m} M_m^2 \\
 &\quad \times (\|\theta_\mu\partial_h^\beta\nabla\mathbf{u}\|_{L^2}^{1/2} \|\theta_\mu\partial_h^{\beta+e_2}\nabla\mathbf{u}\|_{L^2}^{1/2} + \|\theta_\mu\partial_h^{\beta+e_1}\nabla\mathbf{u}\|_{L^2}^{1/2} \|\theta_\mu\partial_h^{\beta+e}\nabla\mathbf{u}\|_{L^2}^{1/2}) \\
 &\lesssim \sum_{m \geq 1} \sum_{j=0}^{[m/2]} \binom{m}{j} \left(\sum_{|\gamma|=m-j} \|\rho^{-1/2}\theta_\mu\partial_h^\gamma\partial_z\mathbf{u}\|_{L^2}^2 \right)^{1/2} \left(\sum_{|\alpha|=m} \|\theta_\mu\partial_h^\alpha\mathbf{u}\|_{L^2}^2 \right)^{1/2} \\
 &\quad \times \sum_{|\beta|=j}^{j+2} \|\theta_\mu\partial_h^\beta\nabla\mathbf{u}\|_{L^2} \tau^{2m} M_m^2 \\
 &\lesssim \sum_{m \geq 1} \sum_{j=0}^{[m/2]} \sum_{i=1}^3 \frac{f_{m,r,j,i} D_{m-j} Y_{j+i} Y_m}{\tau^{i-1}},
 \end{aligned} \tag{3.77}$$

其中, 对于所有的 $0 \leq j \leq [m/2]$ 和 $1 \leq i \leq 3$, 有

$$f_{m,r,j,i} = \frac{\binom{m}{j} M_m}{M_{m-j} M_{j+i} m^{1/2}} \lesssim (j+i)^{i-1/2-r}, \tag{3.78}$$

因此, 当 $r > 3$ 时, 有

$$N_3^{2,1} \lesssim \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{Y_{\tau,\mu}}^2. \tag{3.79}$$

类似地, 对于 $N_3^{2,2}$, 有

$$N_3^{2,2} \lesssim \sum_{m \geq 1} \sum_{j=[m/2]+1}^m \binom{m}{j} \sum_{|\alpha|=m} \sum_{|\beta|=j, \beta \leq \alpha} \|\theta_\mu \partial_h^\beta \nabla \mathbf{u}\|_{L^2} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2$$

$$\begin{aligned}
& \times (\|\rho^{-1/2}\theta_\mu\partial_h^{\alpha-\beta}\partial_z\mathbf{u}\|_{L^2}^{1/2}\|\rho^{-1/2}\theta_\mu\partial_h^{\alpha-\beta+e_2}\partial_z\mathbf{u}\|_{L^2}^{1/2} \\
& + \|\rho^{-1/2}\theta_\mu\partial_h^{\alpha-\beta+e_1}\partial_z\mathbf{u}\|_{L^2}^{1/2}\|\rho^{-1/2}\theta_\mu\partial_h^{\alpha-\beta+e}\partial_z\mathbf{u}\|_{L^2}^{1/2}) \\
& \lesssim \sum_{m \geq 1} \sum_{j=[m/2]+1}^m \binom{m}{j} \left(\sum_{|\beta|=j} \|\theta_\mu\partial_h^\beta \nabla \mathbf{u}\|_{L^2}^2 \right)^{1/2} \left(\sum_{|\alpha|=m} \|\theta_\mu\partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \\
& \times \sum_{|\gamma|=m-j}^{m-j+2} \|\theta_\mu\partial_h^\gamma \partial_z \mathbf{u}\|_{L^2} \tau^{2m} M_m^2 \\
& \lesssim \sum_{m \geq 1} \sum_{j=[m/2]+1}^m \sum_{i=1}^3 \frac{h_{m,r,j,i} Y_{j+1} D_{m-j+i} Y_m}{\tau^{i-1}}. \tag{3.80}
\end{aligned}$$

对 $[m/2] + 1 \leq j \leq m$ 和 $1 \leq i \leq 3$, 有

$$h_{m,r,j,i} = \frac{\binom{m}{j}(m-j+i)^{1/2}M_m}{M_{m-j+i-1}(j+1)^{1/2}M_{j+1}m^{1/2}} \lesssim (m-j+i)^{i-1/2-r}, \tag{3.81}$$

因此, 当 $r > 3$ 时, 可得

$$N_3^{2,2} \lesssim \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{Y_{\tau,\mu}}^2. \tag{3.82}$$

联合 (3.73), (3.79) 和 (3.82), 得到

$$N_3 \lesssim \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\lambda}} \|\mathbf{u}\|_{Y_{\tau,\lambda}} \|\mathbf{u}\|_{X_{\tau,\lambda}} + \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{Y_{\tau,\mu}}^2. \tag{3.83}$$

综合 N_1 - N_3 的估计 (3.58), (3.71) 和 (3.83), 利用 Cauchy 不等式, 可得

$$\begin{aligned}
N & \lesssim \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{X_{\tau,\mu}}^2 + \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\lambda}} \|\mathbf{u}\|_{Y_{\tau,\lambda}} \|\mathbf{u}\|_{X_{\tau,\lambda}} + \frac{1}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{Y_{\tau,\mu}}^2 \\
& \leq \frac{1}{8} \|\mathbf{u}\|_{D_{\tau,\mu}}^2 + \frac{C}{\tau^4} \|\mathbf{u}\|_{X_{\tau,\mu}}^4 + \frac{C}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \|\mathbf{u}\|_{Y_{\tau,\mu}}^2, \tag{3.84}
\end{aligned}$$

则引理 3.3 得证. $\square$

3.3 源项的估计

引理 3.4 给定实数 $\mu > 0$ 和 $\lambda > 1$, 有如下估计:

$$F = \sum_{m \geq 0} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha \mathcal{F}, \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \leq C \|\mathbf{u}\|_{X_{\tau,\mu}}, \tag{3.85}$$

其中 $C = C(C_*, r, \tau_*)$ 为适当的正常数.

证明 由 (2.6), 记

$$\begin{aligned}
\mathcal{F} & = -z\tilde{U}\phi'\mathbf{U} + \phi(1-\phi)\mathbf{U} \cdot \nabla \mathbf{U} - \rho^{-1}\phi''\mathbf{U} - \phi\nabla P \\
& \quad + \Phi(\nabla\tilde{\rho} \cdot \mathbf{U})\phi'\mathbf{U} + \Phi(\nabla \cdot \mathbf{U})\phi'\mathbf{U} \\
& =: \sum_{i=1}^6 \mathcal{F}_i. \tag{3.86}
\end{aligned}$$

令

$$F_i = \sum_{m \geq 0} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha \mathcal{F}_i, \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2, \tag{3.87}$$

则有

$$F = \sum_{m \geq 0} \sum_{|\alpha|=m} |\langle \theta_\mu \partial_h^\alpha \mathcal{F}, \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \leq \sum_{i=1}^6 F_i. \quad (3.88)$$

下面对 F_1-F_6 进行逐项估计.

关于 F_1 , 由 (3.34) 可知, 当 $\lambda > 1$ 时, 有 $z\theta_\mu \phi' \in L_z^2$. 结合不等式 (3.90), 可得

$$\begin{aligned} F_1 &= \sum_{m \geq 0} \sum_{|\alpha|=m} |\langle z\theta_\mu \phi' \partial_h^\alpha (\mathcal{U}\mathbf{U}), \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \\ &\leq \sum_{m \geq 0} \sum_{|\alpha|=m} \|z\theta_\mu \phi'\|_{L_z^2} \|\partial_h^\alpha (\mathcal{U}\mathbf{U})\|_{L_h^2} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2 \\ &\lesssim \sum_{m \geq 0} \left(\sum_{|\alpha|=m} \|\partial_h^\alpha (\mathcal{U}\mathbf{U})\|_{L_h^2}^2 \right)^{1/2} \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^{2m} M_m^2 \\ &\lesssim \sum_{m \geq 0} \left(\frac{\tau}{\tau_*} \right)^m X_m \\ &\lesssim \|\mathbf{u}\|_{X_{\tau,\mu}}. \end{aligned} \quad (3.89)$$

这里用到了 $\tau \leq \tau_*/2$, 且由不等式 (2.17), 结合 Leibniz 公式, 可以证明函数 $\mathcal{U}\mathbf{U}$ 在 $t \in [0, T]$ 上也是实解析的, 且适当增加 C_* 并减小 τ_* , 其满足如下估计:

$$\left(\sum_{|\alpha|=m} \|\partial_h^\alpha (\mathcal{U}\mathbf{U})\|_{L_h^2}^2 \right)^{1/2} \leq \frac{C_*}{\tau_*^m M_m}. \quad (3.90)$$

进一步地, 可以验证 $\mathbf{U} \cdot \nabla \mathbf{U}$, $\rho^{-1} \mathbf{U}$, ∇P , $(\nabla \tilde{\rho} \cdot \mathbf{U})\mathbf{U}$ 和 $(\nabla \cdot \mathbf{U})\mathbf{U}$ 均有上述估计.

对于 F_2 , 有

$$\begin{aligned} F_2 &= \sum_{m \geq 0} \sum_{|\alpha|=m} |\langle \phi(1-\phi) \theta_\mu \partial_h^\alpha (\mathbf{U} \cdot \nabla \mathbf{U}), \theta_\mu \partial_h^\alpha \mathbf{u} \rangle| \tau^{2m} M_m^2 \\ &\leq \sum_{m \geq 0} \sum_{|\alpha|=m} \|\theta_\mu \phi(1-\phi)\|_{L_z^2} \|\partial_h^\alpha (\mathbf{U} \cdot \nabla \mathbf{U})\|_{L_h^2} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2. \end{aligned} \quad (3.91)$$

由于

$$\theta_\mu \phi = \frac{2\sqrt{\lambda}}{\sqrt{\pi}} \int_z^{+\infty} e^{\frac{z^2}{1+t/\mu}} e^{-\lambda\eta^2} d\eta \leq \frac{2\sqrt{\lambda}}{\sqrt{\pi}} \int_z^{+\infty} e^{(\frac{\mu}{1+\mu}-\lambda)\eta^2} d\eta, \quad (3.92)$$

所以当 $\lambda > 1$ 时, 有

$$\|\theta_\mu \phi(1-\phi)\|_{L_z^2} \leq \|1-\phi\|_{L_z^\infty} \|\theta_\mu \phi\|_{L_z^2} \leq C. \quad (3.93)$$

结合不等式 (3.90), 可得

$$\begin{aligned} F_2 &\lesssim \sum_{m \geq 0} \sum_{|\alpha|=m} \|\partial_h^\alpha (\mathbf{U} \cdot \nabla \mathbf{U})\|_{L_h^2} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2 \\ &\lesssim \sum_{m \geq 0} \left(\sum_{|\alpha|=m} \|\partial_h^\alpha (\mathbf{U} \cdot \nabla \mathbf{U})\|_{L_h^2}^2 \right)^{1/2} \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^{2m} M_m^2 \\ &\lesssim \|\mathbf{u}\|_{X_{\tau,\mu}}. \end{aligned} \quad (3.94)$$

下面给出 F_3 的估计:

$$F_3 \leq \sum_{m \geq 0} \sum_{|\alpha|=m} \|\theta_\mu \phi''\|_{L_z^2} \|\partial_h^\alpha (\rho^{-1} \mathbf{U})\|_{L_h^2} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2. \quad (3.95)$$

注意到

$$\theta_\mu \phi'' = \frac{4\lambda\sqrt{\lambda}z}{\sqrt{\pi}} \exp\left[\left(\frac{\mu}{t+\mu} - \lambda\right)z^2\right], \quad (3.96)$$

则当 $\lambda > 1$ 时, 有 $\theta_\mu \phi'' \in L_z^2$, 即得

$$F_3 \lesssim \sum_{m \geq 0} \left(\sum_{|\alpha|=m} \|\partial_h^\alpha (\rho^{-1} \mathbf{U})\|_{L_h^2}^2 \right)^{1/2} \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^{2m} M_m^2 \lesssim \|\mathbf{u}\|_{X_{\tau,\mu}}. \quad (3.97)$$

对于 F_4 , 根据 (3.93), 可得

$$\begin{aligned} F_4 &\leq \sum_{m \geq 0} \sum_{|\alpha|=m} \|\theta_\mu \phi\|_{L_z^2} \|\partial_h^\alpha (\nabla P)\|_{L_h^2} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2 \\ &\lesssim \sum_{m \geq 0} \left(\sum_{|\alpha|=m} \|\partial_h^\alpha (\nabla P)\|_{L_h^2}^2 \right)^{1/2} \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^{2m} M_m^2 \\ &\lesssim \|\mathbf{u}\|_{X_{\tau,\mu}}. \end{aligned} \quad (3.98)$$

接下来给出 F_5 的估计, 利用 (3.34) 和 Hölder 不等式, 有

$$\begin{aligned} F_5 &\leq \sum_{m \geq 0} \sum_{|\alpha|=m} \|\Phi\|_{L_z^\infty} \|\theta_\mu \phi'\|_{L_z^2} \|\partial_h^\alpha ((\nabla \tilde{\rho} \cdot \mathbf{U}) \mathbf{U})\|_{L_h^2} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2 \\ &\lesssim \sum_{m \geq 0} \left(\sum_{|\alpha|=m} \|\partial_h^\alpha ((\nabla \tilde{\rho} \cdot \mathbf{U}) \mathbf{U})\|_{L_h^2}^2 \right)^{1/2} \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^{2m} M_m^2 \\ &\lesssim \|\mathbf{u}\|_{X_{\tau,\mu}}. \end{aligned} \quad (3.99)$$

类似地, 有

$$\begin{aligned} F_6 &\leq \sum_{m \geq 0} \sum_{|\alpha|=m} \|\Phi\|_{L_z^\infty} \|\theta_\mu \phi'\|_{L_z^2} \|\partial_h^\alpha ((\nabla \cdot \mathbf{U}) \mathbf{U})\|_{L_h^2} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2} \tau^{2m} M_m^2 \\ &\lesssim \sum_{m \geq 0} \left(\sum_{|\alpha|=m} \|\partial_h^\alpha ((\nabla \cdot \mathbf{U}) \mathbf{U})\|_{L_h^2}^2 \right)^{1/2} \left(\sum_{|\alpha|=m} \|\theta_\mu \partial_h^\alpha \mathbf{u}\|_{L^2}^2 \right)^{1/2} \tau^{2m} M_m^2 \\ &\lesssim \|\mathbf{u}\|_{X_{\tau,\mu}}. \end{aligned} \quad (3.100)$$

综合 F_1 – F_6 的估计, 有

$$F \lesssim \|\mathbf{u}\|_{X_{\tau,\mu}}. \quad (3.101)$$

引理 3.4 证毕. $\square$

3.4 命题 3.1 的证明

综合 (3.12) 和引理 3.2–3.4, 有如下估计:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|_{X_{\tau,\mu}}^2 + (\mu - C\mu^2) \|\mathbf{u}\|_{Z_{\tau,\mu}}^2 + \frac{1}{8} \|\mathbf{u}\|_{D_{\tau,\mu}}^2 \\ & \leq \left(\tau' + C + \frac{C}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} \right) \|\mathbf{u}\|_{Y_{\tau,\mu}}^2 + C \left(\frac{1}{\tau^4} \|\mathbf{u}\|_{X_{\tau,\mu}}^4 + \left(1 + \frac{1}{\tau} \right) \|\mathbf{u}\|_{X_{\tau,\mu}}^2 + \|\mathbf{u}\|_{X_{\tau,\mu}} \right). \end{aligned} \quad (3.102)$$

令解析半径 $\tau(t)$ 满足如下的初值问题:

$$\begin{cases} \dot{\tau} + C + \frac{C}{\tau^2} \|\mathbf{u}\|_{D_{\tau,\mu}} = 0, \\ \tau(0) = \tau_0. \end{cases} \quad (3.103)$$

显然 $\tau(t)$ 是单调递减的, 且满足

$$\tau^3 \geq \tau_0^3 - 3C\tau_0^2 t - 3Ct^{1/2} \left(\int_0^t \|\mathbf{u}(s)\|_{D_{\tau,\mu}}^2 ds \right)^{1/2}. \quad (3.104)$$

假设对于适当小的时间 $0 < T_1 < T$, 有

$$\int_0^t \|\mathbf{u}(s)\|_{D_{\tau,\mu}}^2 ds \leq C \|\mathbf{u}_0\|_{X_{\tau_0,\mu}}^2, \quad (3.105)$$

则由 (3.104), 存在时间 $0 < T_2 < T_1$, 使得

$$\frac{\tau_0}{3} \leq \tau(t). \quad (3.106)$$

最后, 验证 (3.105). 令

$$\mu - C\mu^2 \geq 0 \rightarrow \mu \leq 1/C, \quad (3.107)$$

再结合 (3.102), (3.103) 和 (3.106), 有

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|_{X_{\tau,\mu}}^2 + \frac{1}{8} \|\mathbf{u}\|_{D_{\tau,\mu}}^2 \lesssim \|\mathbf{u}\|_{X_{\tau,\mu}}^4 + \|\mathbf{u}\|_{X_{\tau,\mu}}^2 + \|\mathbf{u}\|_{X_{\tau,\mu}}. \quad (3.108)$$

进一步地, 根据常微分方程比较原理, 存在足够小的时间 T_* , 有

$$\|\mathbf{u}\|_{X_{\tau,\mu}}^2 + \int_0^t \|\mathbf{u}(s)\|_{D_{\tau,\mu}}^2 ds \lesssim \|\mathbf{u}_0\|_{X_{\tau_0,\mu}}^2, \quad (3.109)$$

即 (3.105) 成立, 命题 3.1 得证. $\square$

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The local well-posedness of the three-dimensional compressible isentropic boundary layer equations

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Abstract The Prandtl equations governing the boundary layer for incompressible flows are extensively studied in different contexts. The mathematical study of boundary layers for compressible flows remains relatively limited. In particular, in the three-dimensional setting, there are few results for boundary layer problems in either incompressible or compressible flows. In this paper, we consider the three-dimensional boundary layer equations for compressible isentropic flows with no-slip boundary conditions. Under the assumption that the initial data is analytic in the tangential variables and has Sobolev regularity in the normal direction, we establish the local well-posedness of the system using direct energy estimates.

Keywords boundary layer equation, three-dimensional compressible flow, exponential weight, well-posedness, energy method

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