



快慢 McKean-Vlasov 随机微分方程的大偏差原理

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摘要 本文研究当噪声强度 $\delta \rightarrow 0$ 且时间尺度参数 $\varepsilon(\delta)$ 满足 $\varepsilon^2(\delta)/\delta \rightarrow 0$ 时, 一类完全耦合的快慢 McKean-Vlasov 随机系统解的 Freidlin-Wentzell 型大偏差原理. 该系统系数具有对慢、快变量分布的双重依赖. 证明的核心在于借助 Poisson 方程理论及大偏差的弱收敛方法.

关键词 大偏差原理 快慢 McKean-Vlasov 随机微分方程 Poisson 方程 弱收敛方法

MSC (2020) 主题分类 60F10, 60H10, 60J60

1 引言

多尺度随机系统已广泛应用于气象、金融和生物等多个领域. 考虑如下快慢 McKean-Vlasov 随机微分方程 (MVSDE):

$$\begin{cases} dX_t^\varepsilon = b(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon})dt + \sigma(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, \mathcal{L}_{Y_t^\varepsilon})dW_t^1, & X_0^\varepsilon = x \in \mathbb{R}^{d_1}, \\ dY_t^\varepsilon = \frac{1}{\varepsilon^2}f(\mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon})dt + \frac{1}{\varepsilon}g(\mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon})dW_t^2, & Y_0^\varepsilon = y \in \mathbb{R}^{d_2}, \end{cases} \quad (1.1)$$

其中时间尺度参数 $0 < \varepsilon \ll 1$, $b: \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}) \rightarrow \mathbb{R}^{d_1}$, $\sigma: \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathcal{P}_2(\mathbb{R}^{d_2}) \rightarrow \mathbb{R}^{d_1} \otimes \mathbb{R}^{d_1}$, $f: \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}) \rightarrow \mathbb{R}^{d_2}$, $g: \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}) \rightarrow \mathbb{R}^{d_2} \otimes \mathbb{R}^{d_2}$ 均为 Borel 可测函数; W_t^1 和 W_t^2 分别为定义在完备概率空间 $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t \geq 0})$ 上的 d_1 和 d_2 维 Brown 运动, 且二者相互独立; $\mathcal{L}_\vartheta$ 表示随机变量 ϑ 的分布.

与此前大多数关于快慢 MVSDE 的研究相比 (参见文献 [1, 2, 4]), 系统 (1.1) 的主要特点在于, 方程的系数不仅依赖于慢变量的分布, 同时还依赖于快变量的分布, 这一特性为相关问题的研究带

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来了极大的挑战. 为刻画系统 (1.1) 的平均化极限, 需考虑如下的凝固 MVSDE: 对任意固定的 $\mu \in \mathcal{P}_2(\mathbb{R}^{d_1})$, 有

$$dY_t^\mu = f(\mu, Y_t^\mu, \mathcal{L}_{Y_t^\mu})dt + g(\mu, Y_t^\mu, \mathcal{L}_{Y_t^\mu})d\tilde{W}_t^2, \quad Y_0^\mu = y, \quad (1.2)$$

其中 $\tilde{W}_t^2 = \varepsilon^{-1}W_{\varepsilon^2 t}^2$ 构成一个新的 d_2 维 Brown 运动. 假设存在常数 $c_2 > c_1 \geq 0$, 使得对任意的 $\mu \in \mathcal{P}_2(\mathbb{R}^{d_1})$, $y_1, y_2 \in \mathbb{R}^{d_2}$ 及 $\nu_1, \nu_2 \in \mathcal{P}_2(\mathbb{R}^{d_2})$, 有

$$\begin{aligned} & 2\langle y_1 - y_2, f(\mu, y_1, \nu_1) - f(\mu, y_2, \nu_2) \rangle + 3\|g(\mu, y_1, \nu_1) - g(\mu, y_2, \nu_2)\|^2 \\ & \leq c_1 \mathbb{W}_2(\nu_1, \nu_2)^2 - c_2 |y_1 - y_2|^2. \end{aligned} \quad (1.3)$$

由文献 [9, 定理 3.1] 可知, 方程 (1.2) 存在唯一的不变测度 ζ^μ . 最近, Li 和 Xie [6] 证明了当 $\varepsilon \rightarrow 0$ 时, 上述系统中慢变量 X_t^ε 强收敛到 $\bar{X}_t$, 即

$$\sup_{t \in [0, T]} \mathbb{E}|X_t^\varepsilon - \bar{X}_t|^2 \leq C_T \varepsilon^2,$$

其中 $\bar{X}_t$ 满足平均化方程

$$d\bar{X}_t = \bar{b}(\bar{X}_t, \mathcal{L}_{\bar{X}_t})dt + \bar{\sigma}(\bar{X}_t, \mathcal{L}_{\bar{X}_t})dW_t^1, \quad \bar{X}_0 = x, \quad (1.4)$$

相应的平均化系数定义为

$$\bar{b}(x, \mu) := \int_{\mathbb{R}^{d_2}} b(x, \mu, y, \zeta^\mu) \zeta^\mu(dy), \quad \bar{\sigma}(x, \mu) := \sigma(x, \mu, \zeta^\mu). \quad (1.5)$$

当前, 多尺度随机系统的大偏差研究主要集中于非分布依赖情形, 而针对多尺度 MVSDE 大偏差的研究尚处于起步阶段. Hong 等 [4] 研究了一类多尺度 MVSDE 的大偏差原理, 其中快方程系数可依赖于慢过程, 但系统整体不依赖于快变量的分布. 随后, 文献 [8, 10] 进一步探讨了带有分数噪声的快慢 MVSDE 的大偏差问题, 其系统系数同样不依赖于快变量的分布. 与之相比, MVSDE (1.1) 具有更强的分布依赖性. 本文基于 Li 和 Xie [6] 的研究, 运用 Dupuis 和 Ellis [3] 提出的弱收敛方法, 建立其慢变量在小噪声扰动下满足的 Freidlin-Wentzell 型大偏差原理. 与文献 [4, 8, 10] 采用的离散化时间方法不同, 本文充分运用 Poisson 方程的处理技巧建立大偏差原理.

2 预备知识及主要结论

本节阐述运用弱收敛方法证明大偏差原理时涉及的相关定义与关键结果. 对文中所涉及的函数空间作出明确界定, 同时给出主要研究结果.

2.1 预备知识

令 $\mathcal{P}(\mathbb{R}^d)$ 表示 $\mathbb{R}^d$ 上的概率测度全体, 并定义

$$\mathcal{P}_2(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \mu(|\cdot|^2) := \int_{\mathbb{R}^d} |\xi|^2 \mu(d\xi) < \infty \right\}.$$

对任意 $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, 记 $\mathcal{C}(\mu, \nu)$ 为 μ 和 ν 的耦合测度全体. 2 阶 Wasserstein 距离定义为

$$\mathbb{W}_2(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu, \nu)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dx, dy) \right)^{\frac{1}{2}}.$$

对任意给定 $T > 0$, $C([0, T]; \mathbb{R}^{d_1})$ 表示 $[0, T]$ 上 $\mathbb{R}^{d_1}$ 值连续函数全体. 对任一 $\psi \in C([0, T]; \mathbb{R}^{d_1})$, 定义 $\|\psi\|_\infty := \sup_{t \in [0, T]} |\psi(t)|$, d_∞ 表示由该范数诱导的度量, 则 $(C([0, T]; \mathbb{R}^{d_1}), d_\infty)$ 为 Polish 空间.

令 $W_t = (W_t^1, W_t^2)$, 记 $P_1: \mathbb{R}^{d_1+d_2} \rightarrow \mathbb{R}^{d_1}$ 和 $P_2: \mathbb{R}^{d_1+d_2} \rightarrow \mathbb{R}^{d_2}$ 为投影算子, 使得

$$W_t^1 = P_1 W_t, \quad W_t^2 = P_2 W_t.$$

考虑与过程 $(W_t)_{0 \leq t \leq T}$ 相关联的 Cameron-Martin 空间

$$\mathbb{H} := \left\{ h: [0, T] \rightarrow \mathbb{R}^{d_1+d_2} \mid h_0 = 0, h \text{ 绝对连续}, \|h\|_{\mathbb{H}}^2 := \int_0^T |\dot{h}_s|^2 ds < \infty \right\}.$$

对任意 $N > 0$, 定义

$$B_N := \{h \in \mathbb{H} : \|h\|_{\mathbb{H}} \leq N\}.$$

在 $\mathbb{H}$ 诱导的弱拓扑下, B_N 构成紧 Polish 空间. 本文关于取值于 B_N 的随机变量的收敛性, 均在此拓扑下考虑. 令 $\mathcal{A}$ 为全体 $\mathcal{F}_t$ -可料过程, 并记

$$\mathcal{A}_N := \{h \in \mathcal{A} : h \in B_N, \mathbb{P}\text{-a.s.}\}.$$

设 (E, d) 为 Polish 空间, $\mathcal{B}(E)$ 为其 Borel σ -域. $\{X^\delta, \delta > 0\}$ 为定义于概率空间 $(\Omega, \mathcal{F}, \mathbb{P})$ 上的一族 E 值随机变量.

定义 2.1 对任意 $a < \infty$, 若 $\{\phi \in E : I(\phi) \leq a\}$ 是紧集, 则称 $I: E \rightarrow [0, \infty]$ 为速率函数. 对任意 $A \in \mathcal{B}(E)$, 记 $I(A) := \inf_{\phi \in A} I(\phi)$.

定义 2.2 设 I 为 E 上的速率函数.

(1) 称 $\{X^\delta, \delta > 0\}$ 满足速率函数为 I 的大偏差上界, 若对 E 中的一切闭子集 F , 有

$$\overline{\lim}_{\delta \rightarrow 0} \delta \log \mathbb{P}(X^\delta \in F) \leq -I(F);$$

(2) 称 $\{X^\delta, \delta > 0\}$ 满足速率函数为 I 的大偏差下界, 若对 E 中的一切开子集 G , 有

$$\underline{\lim}_{\delta \rightarrow 0} \delta \log \mathbb{P}(X^\delta \in G) \geq -I(G).$$

若大偏差上界和下界同时满足, 则称 $\{X^\delta, \delta > 0\}$ 满足速率函数为 I 的大偏差原理.

相较于开集或闭集的示性函数估计, 研究者们更关注一般函数的指数估计. 回顾文献 [7, 11] 中的相关结果, 下面直接给出 Laplace 原理成立的充分条件.

定理 2.1 对任意 $\delta > 0$, 设 $Z^\delta: C([0, T]; \mathbb{R}^{d_1+d_2}) \rightarrow E$ 为一族可测映射, 并记 $X^\delta = Z^\delta(W)$. 若存在可测映射 $Z^0: C([0, T]; \mathbb{R}^{d_1+d_2}) \rightarrow E$, 使得以下条件成立:

(LD₁) 对任意 $N > 0$, 若当 $\delta \rightarrow 0$ 时, h^δ 在 B_N 中弱收敛于 h , 则 $Z^0(\int_0^\cdot \dot{h}_s^\delta ds)$ 在空间 E 中收敛于 $Z^0(\int_0^\cdot \dot{h}_s ds)$;

(LD₂) 对任意 $N > 0$, 若 $\{h^\delta\}_{\delta>0} \subset \mathcal{A}_N$ (作为取值于 B_N 中的随机变量), 则对任意 $\eta > 0$, 有

$$\lim_{\delta \rightarrow 0} \mathbb{P} \left(d \left(Z^\delta \left(W + \frac{1}{\sqrt{\delta}} \int_0^\cdot \dot{h}_s^\delta ds \right), Z^0 \left(\int_0^\cdot \dot{h}_s^\delta ds \right) \right) > \eta \right) = 0,$$

则 $\{X^\delta, \delta > 0\}$ 在空间 E 上满足 Laplace 原理, 对应速率函数为

$$I(\phi) := \frac{1}{2} \inf_{\{h \in \mathbb{H}: Z^0(\int_0^\cdot \dot{h}_s ds) = \phi\}} \|h\|_{\mathbb{H}}^2, \quad (2.1)$$

其中约定 $\inf\{\emptyset\} = \infty$, 即对 E 上所有有界连续函数 φ , 都有

$$\lim_{\delta \rightarrow 0} \delta \log \mathbb{E} \left(\exp \left[- \frac{\varphi(X^\delta)}{\delta} \right] \right) = - \inf_{\phi \in E} \{ \varphi(\phi) + I(\phi) \}.$$

根据 Laplace 原理与大偏差原理的等价性结果 (参见文献 [3]), 可得如下结论.

定理 2.2 在定理 2.1 的条件下, $\{X^\delta, \delta > 0\}$ 在 E 上满足具有相同速率函数 I 的大偏差原理.

2.2 主要结论

在介绍本文核心结论之前, 我们需要对定理中涉及的函数空间予以严格定义.

定义 2.3 若定义在 $\mathcal{P}_2(\mathbb{R}^d)$ 上的函数 $u(\cdot)$, 其 Lions 导数 $\partial_\mu u(\mu)(\tilde{x})$ 及其空间导数 $\partial_{\tilde{x}} \partial_\mu u(\mu)(\tilde{x})$ 均存在且连续, 并满足一致有界条件 $\sup_{\mu \in \mathcal{P}_2, \tilde{x} \in \mathbb{R}^d} |\partial_\mu u(\mu)(\tilde{x})| < \infty$ 和 $\sup_{\mu \in \mathcal{P}_2, \tilde{x} \in \mathbb{R}^d} \|\partial_{\tilde{x}} \partial_\mu u(\mu)(\tilde{x})\| < \infty$, 则称 $u \in C^{(1,1)}(\mathcal{P}_2(\mathbb{R}^d))$. $C^2(\mathbb{R}^d)$ 表示 $\mathbb{R}^d$ 上具有有界连续一阶及二阶偏导数的函数空间. 若定义在 $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ 上的函数 $u(\cdot, \cdot)$ 满足, 对任意 $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, 有 $u(\cdot, \mu) \in C^2(\mathbb{R}^d)$; 且对任意 $x \in \mathbb{R}^d$, 有 $u(x, \cdot) \in C^{(1,1)}(\mathcal{P}_2(\mathbb{R}^d))$, 则称 $u \in C^{2,(1,1)}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$. 若定义在 $\mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2})$ 上的函数 $u(\cdot, \cdot, \cdot)$ 满足, 对任意 $(y, \nu) \in \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2})$, 有 $u(\cdot, y, \nu) \in C^{(1,1)}(\mathcal{P}_2(\mathbb{R}^{d_1}))$; 且对任意 $\mu \in \mathcal{P}_2(\mathbb{R}^{d_1})$, 有 $u(\mu, \cdot, \cdot) \in C^{2,(1,1)}(\mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$, 则称 $u \in C^{(1,1),2,(1,1)}(\mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$. 若定义在 $\mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2})$ 上的函数 $u(\cdot, \cdot, \cdot, \cdot)$ 满足, 对任意 $(\mu, y, \nu) \in \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2})$, 有 $u(\cdot, \mu, y, \nu) \in C^2(\mathbb{R}^{d_1})$; 且对任意 $x \in \mathbb{R}^{d_1}$, 有 $u(x, \cdot, \cdot, \cdot) \in C^{(1,1),2,(1,1)}(\mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$, 则称 $u \in C^{2,(1,1),2,(1,1)}(\mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$. 若上述空间添加下标 b , 则表示函数本身也是有界的.

在 MVSDE (1.1) 的慢变量扩散项中引入噪声强度参数 $0 < \delta < 1$, 得到如下小扰动系统:

$$\begin{cases} dX_t^{\delta,\varepsilon} = b(X_t^{\delta,\varepsilon}, \mathcal{L}_{X_t^{\delta,\varepsilon}}, Y_t^{\delta,\varepsilon}, \mathcal{L}_{Y_t^{\delta,\varepsilon}})dt + \sqrt{\delta}\sigma(X_t^{\delta,\varepsilon}, \mathcal{L}_{X_t^{\delta,\varepsilon}}, \mathcal{L}_{Y_t^{\delta,\varepsilon}})dW_t^1, & X_0^{\delta,\varepsilon} = x, \\ dY_t^{\delta,\varepsilon} = \frac{1}{\varepsilon^2}f(\mathcal{L}_{X_t^{\delta,\varepsilon}}, Y_t^{\delta,\varepsilon}, \mathcal{L}_{Y_t^{\delta,\varepsilon}})dt + \frac{1}{\varepsilon}g(\mathcal{L}_{X_t^{\delta,\varepsilon}}, Y_t^{\delta,\varepsilon}, \mathcal{L}_{Y_t^{\delta,\varepsilon}})dW_t^2, & Y_0^{\delta,\varepsilon} = y. \end{cases} \quad (2.2)$$

本文主要结论如下.

定理 2.3 假设条件 (1.3) 成立, 且 $b \in C^{2,(1,1),2,(1,1)}(\mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$, $f \in C^{(1,1),2,(1,1)}(\mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$, $g \in C_b^{(1,1),2,(1,1)}(\mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$, b, σ, f 和 g 满足 Lipschitz 条件并且

$$\sup_{\mu \in \mathcal{P}_2(\mathbb{R}^{d_1})} |f(\mu, 0, \delta_0)| < \infty, \quad \sup_{x \in \mathbb{R}^{d_1}} \|\sigma(x, \delta_0, \delta_0)\| < \infty.$$

当 $\lim_{\delta \rightarrow 0} \varepsilon^2/\delta = 0$ 时, 扰动系统 (2.2) 的慢过程族 $\{X^{\delta,\varepsilon}, \delta > 0\}$ 在 $C([0, T]; \mathbb{R}^{d_1})$ 上满足大偏差原理, 其速率函数由 (2.1) 给出, 其中 Z^0 由后续的 (3.6) 定义.

3 定理 2.3 的证明

本节始终假设定理 2.3 中的条件成立. 针对 MVSDE (2.2), 依据平均原理 (参见文献 [6, 定理 2.1]), 当 $\delta \rightarrow 0$ 时, 其解收敛于确定性平均化方程

$$d\bar{X}_t^0 = \bar{b}(\bar{X}_t^0, \mathcal{L}_{\bar{X}_t^0})dt, \quad \bar{X}_0^0 = x, \quad (3.1)$$

其中平均化系数 $\bar{b}$ 通过 (1.5) 定义.

注 3.1 由于 $\bar{X}_t^0$ 是确定性轨道, 所以 $\mathcal{L}_{\bar{X}_t^0} = \delta_{\bar{X}_t^0}$, 其中 $\delta_{\bar{X}_t^0}$ 表示集中于点 $\bar{X}_t^0$ 的 Dirac 测度. 由于 b 满足 Lipschitz 条件, 结合文献 [6, 引理 4.2] 的结论, 可推导出 $\bar{b}$ 也满足 Lipschitz 条件. 显然方程 (3.1) 存在唯一解 $\bar{X}^0$ 且 $\bar{X}^0 \in C([0, T]; \mathbb{R}^{d_1})$.

为验证 (LD₁), 考虑骨架方程

$$d\bar{X}_t^h = \bar{b}(\bar{X}_t^h, \mathcal{L}_{\bar{X}_t^h})dt + \bar{\sigma}(\bar{X}_t^h, \mathcal{L}_{\bar{X}_t^h})P_1 \dot{h}_t dt, \quad \bar{X}_0^h = x, \quad (3.2)$$

其中 $\bar{X}_t^0$ 是平均化方程 (3.1) 的唯一解且平均化系数 $\bar{b}$ 和 $\bar{\sigma}$ 由 (1.5) 定义.

下面证明骨架方程 (3.2) 解的存在唯一性, 同时给出相应的估计.

引理 3.1 对任意 $h \in \mathbb{H}$, 骨架方程 (3.2) 存在唯一绝对连续解 $\bar{X}^h$, 且

$$\sup_{h \in B_N} \sup_{t \in [0, T]} |\bar{X}_t^h| \leq C_{N, T}(1 + |x|). \quad (3.3)$$

证明 由文献 [6, 引理 4.2] 可知, $\bar{b}$ 和 $\bar{\sigma}$ 均满足 Lipschitz 条件, 即

$$|\bar{b}(x_1, \mu_1) - \bar{b}(x_2, \mu_2)| + \|\bar{\sigma}(x_1, \mu_1) - \bar{\sigma}(x_2, \mu_2)\| \leq C(|x_1 - x_2| + \mathbb{W}_2(\mu_1, \mu_2)). \quad (3.4)$$

对固定测度 $\mathcal{L}_{\bar{X}_t^0}$, 记 $\tilde{b}(t, x) := \bar{b}(x, \mathcal{L}_{\bar{X}_t^0})$, $\tilde{\sigma}(t, x) := \bar{\sigma}(x, \mathcal{L}_{\bar{X}_t^0})$, 则

$$|\tilde{b}(t, x_1) - \tilde{b}(t, x_2)| + \|\tilde{\sigma}(t, x_1) - \tilde{\sigma}(t, x_2)\| \leq C|x_1 - x_2|.$$

于是方程 (3.2) 可改写为

$$d\bar{X}_t^h = \tilde{b}(t, \bar{X}_t^h)dt + \tilde{\sigma}(t, \bar{X}_t^h)P_1 \dot{h}_t dt, \quad \bar{X}_0^h = x. \quad (3.5)$$

通过 Picard 迭代法可证明该方程存在唯一绝对连续解. 基于线性增长条件和 Gronwall 不等式, 结合 $\bar{X}^0 \in C([0, T]; \mathbb{R}^{d_1})$, 可进一步得到解的先验估计 (3.3). $\square$

对任意 $h \in \mathbb{H}$, 定义映射

$$Z^0 : \mathbb{H} \ni h \mapsto \bar{X}^h \in C([0, T]; \mathbb{R}^{d_1}), \quad (3.6)$$

其中 $\bar{X}^h$ 为骨架方程 (3.2) 的唯一解. 则 (2.1) 可改写为

$$I(\phi) = \frac{1}{2} \inf_{\{h \in \mathbb{H} : \bar{X}^h = \phi\}} \|h\|_{\mathbb{H}}^2. \quad (3.7)$$

接下来验证 (LD₁), 即证明 I 是 $C([0, T]; \mathbb{R}^{d_1})$ 上的速率函数.

引理 3.2 若当 $\delta \rightarrow 0$ 时, h^δ 在 B_N 中弱收敛到 h , 则有

$$\lim_{\delta \rightarrow 0} \sup_{t \in [0, T]} \left| Z^0 \left(\int_0^t \dot{h}_s^\delta ds \right) - Z^0 \left(\int_0^t \dot{h}_s ds \right) \right| = 0. \quad (3.8)$$

证明 令

$$\bar{X}^{h^\delta} = Z^0 \left(\int_0^\cdot \dot{h}_s^\delta ds \right),$$

则 $\bar{X}_t^{h^\delta}$ 为如下方程的唯一解:

$$d\bar{X}_t^{h^\delta} = \bar{b}(\bar{X}_t^{h^\delta}, \mathcal{L}_{\bar{X}_t^{h^\delta}})dt + \bar{\sigma}(\bar{X}_t^{h^\delta}, \mathcal{L}_{\bar{X}_t^{h^\delta}})P_1 \dot{h}_t^\delta dt, \quad \bar{X}_0^{h^\delta} = x. \quad (3.9)$$

由于 $\bar{b}$ 和 $\bar{\sigma}$ 均满足 Lipschitz 条件, 验证

$$\lim_{\delta \rightarrow 0} \sup_{t \in [0, T]} |\bar{X}_t^{h^\delta} - \bar{X}_t^h| = 0$$

的过程是标准的, 所以不再重复. $\square$

下文旨在验证 (LD₂) 成立. 回到扰动方程 (2.2). 由于 b, σ, f 和 g 均满足 Lipschitz 条件, 则 MVSDE (2.2) 存在唯一的强解 $(X^{\delta, \varepsilon}, Y^{\delta, \varepsilon}) \in C([0, T]; \mathbb{R}^{d_1 + d_2})$. 记

$$\begin{aligned} \tilde{b}^{\delta, \varepsilon}(t, x, y) &:= b(x, \mathcal{L}_{X_t^{\delta, \varepsilon}}, y, \mathcal{L}_{Y_t^{\delta, \varepsilon}}), \quad \tilde{\sigma}^{\delta, \varepsilon}(t, x) := \sigma(x, \mathcal{L}_{X_t^{\delta, \varepsilon}}, \mathcal{L}_{Y_t^{\delta, \varepsilon}}), \\ \tilde{f}^{\delta, \varepsilon}(t, y) &:= f(\mathcal{L}_{X_t^{\delta, \varepsilon}}, y, \mathcal{L}_{Y_t^{\delta, \varepsilon}}), \quad \tilde{g}^{\delta, \varepsilon}(t, y) := g(\mathcal{L}_{X_t^{\delta, \varepsilon}}, y, \mathcal{L}_{Y_t^{\delta, \varepsilon}}), \end{aligned}$$

则 MVSDE (2.2) 可改写为以下非时齐经典随机微分方程 (SDE):

$$\begin{cases} dX_t^{\delta, \varepsilon} = \tilde{b}^{\delta, \varepsilon}(t, X_t^{\delta, \varepsilon}, Y_t^{\delta, \varepsilon})dt + \sqrt{\delta} \tilde{\sigma}^{\delta, \varepsilon}(t, X_t^{\delta, \varepsilon})dW_t^1, & X_0^{\delta, \varepsilon} = x, \\ dY_t^{\delta, \varepsilon} = \frac{1}{\varepsilon^2} \tilde{f}^{\delta, \varepsilon}(t, Y_t^{\delta, \varepsilon})dt + \frac{1}{\varepsilon} \tilde{g}^{\delta, \varepsilon}(t, Y_t^{\delta, \varepsilon})dW_t^2, & Y_0^{\delta, \varepsilon} = y. \end{cases} \quad (3.10)$$

由经典 Yamada-Watanabe 定理可知, 存在唯一可测泛函 $\Gamma^{\delta, \varepsilon} : C([0, T]; \mathbb{R}^{d_1 + d_2}) \rightarrow C([0, T]; \mathbb{R}^{d_1})$ 使得 $X^{\delta, \varepsilon} = \Gamma^{\delta, \varepsilon}(W)$, $\mathbb{P}$ -a.s. 对固定的 $\delta \in (0, 1)$, $h^\delta \in \mathcal{A}_N$, 考虑漂移变换

$$T_\delta(W) := W + \frac{1}{\sqrt{\delta}} \int_0^\cdot \dot{h}_s^\delta ds.$$

根据 Girsanov 定理, 可知测度 $\mathbb{P} \circ (T_\delta)^{-1}$ 与 $\mathbb{P}$ 等价且 $W_t + \frac{1}{\sqrt{\delta}} \int_0^t \dot{h}_s^\delta ds$ 是新测度 $\mathbb{P} \circ (T_\delta)^{-1}$ 下的 Brown 运动. 令 $Z^\delta := \Gamma^{\delta, \varepsilon(\delta)}$, 定义

$$X^{\delta, \varepsilon, h^\delta} := Z^\delta \left(W + \frac{1}{\sqrt{\delta}} \int_0^\cdot \dot{h}_s^\delta ds \right).$$

则 $X_t^{\delta, \varepsilon, h^\delta}$ 满足如下受控系统的慢方程:

$$\begin{cases} dX_t^{\delta, \varepsilon, h^\delta} = b(X_t^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{X_t^{\delta, \varepsilon}}, Y_t^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{Y_t^{\delta, \varepsilon}})dt + \sigma(X_t^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{X_t^{\delta, \varepsilon}}, \mathcal{L}_{Y_t^{\delta, \varepsilon}})P_1 \dot{h}_t^\delta dt, \\ \quad + \sqrt{\delta} \sigma(X_t^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{X_t^{\delta, \varepsilon}}, \mathcal{L}_{Y_t^{\delta, \varepsilon}})dW_t^1, \quad X_0^{\delta, \varepsilon, h^\delta} = x, \\ dY_t^{\delta, \varepsilon, h^\delta} = \frac{1}{\varepsilon^2} f(\mathcal{L}_{X_t^{\delta, \varepsilon}}, Y_t^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{Y_t^{\delta, \varepsilon}})dt + \frac{1}{\sqrt{\delta} \varepsilon} g(\mathcal{L}_{X_t^{\delta, \varepsilon}}, Y_t^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{Y_t^{\delta, \varepsilon}})P_2 \dot{h}_t^\delta dt \\ \quad + \frac{1}{\varepsilon} g(\mathcal{L}_{X_t^{\delta, \varepsilon}}, Y_t^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{Y_t^{\delta, \varepsilon}})dW_t^2, \quad Y_0^{\delta, \varepsilon, h^\delta} = y. \end{cases} \quad (3.11)$$

根据文献 [6, 引理 3.3], 可知扰动方程 (2.2) 的解 $(X^{\delta, \varepsilon}, Y^{\delta, \varepsilon})$ 满足如下先验估计.

引理 3.3 对任意 $T > 0$, 存在常数 $C_T > 0$, 使得

$$\sup_{0 < \varepsilon \ll 1} \mathbb{E} \left[\sup_{t \in [0, T]} |X_t^{\delta, \varepsilon}|^4 \right] \leq C_T (1 + |x|^4 + |y|^4),$$

$$\sup_{0 < \varepsilon \ll 1} \sup_{t \in [0, T]} \mathbb{E} |Y_t^{\delta, \varepsilon}|^4 \leq C_T (1 + |x|^4 + |y|^4).$$

下述估计是证明 (LD₂) (即引理 3.8) 的关键依据. 尽管当 $\delta \rightarrow 0$ 时, (3.12) 的右端项会趋于无穷, 但在后续应用此引理时, 该估计的精度足以支撑后续证明.

引理 3.4 对任意 $T > 0$, 存在不依赖于 T 的常数 $C_1 > 0$, 使得

$$\mathbb{E} \left[\sup_{t \in [0, T]} |Y_t^{\delta, \varepsilon}|^2 \right] \leq C_1 \left(1 + \frac{T}{\varepsilon^2} \right). \quad (3.12)$$

证明 设 $\hat{Y}_t^{\delta, \varepsilon} := Y_{\varepsilon^2 t}^{\delta, \varepsilon}$, 则

$$d\hat{Y}_t^{\delta, \varepsilon} = f(\mathcal{L}_{X_{\varepsilon^2 t}^{\delta, \varepsilon}}, \hat{Y}_t^{\delta, \varepsilon}, \mathcal{L}_{\hat{Y}_t^{\delta, \varepsilon}})dt + g(\mathcal{L}_{X_{\varepsilon^2 t}^{\delta, \varepsilon}}, \hat{Y}_t^{\delta, \varepsilon}, \mathcal{L}_{\hat{Y}_t^{\delta, \varepsilon}})d\tilde{W}_t^2, \quad \hat{Y}_0^{\delta, \varepsilon} = y,$$

其中 $\tilde{W}_t^2 = \varepsilon^{-1} W_{\varepsilon^2 t}^2$. 由 Itô 公式可知

$$\begin{aligned} |\hat{Y}_t^{\delta, \varepsilon}|^2 &= |y|^2 + 2 \int_0^t \langle \hat{Y}_s^{\delta, \varepsilon}, f(\mathcal{L}_{X_{\varepsilon^2 s}^{\delta, \varepsilon}}, \hat{Y}_s^{\delta, \varepsilon}, \mathcal{L}_{\hat{Y}_s^{\delta, \varepsilon}}) \rangle ds + \int_0^t \|g(\mathcal{L}_{X_{\varepsilon^2 s}^{\delta, \varepsilon}}, \hat{Y}_s^{\delta, \varepsilon}, \mathcal{L}_{\hat{Y}_s^{\delta, \varepsilon}})\|^2 ds \\ &\quad + 2 \int_0^t \langle \hat{Y}_s^{\delta, \varepsilon}, g(\mathcal{L}_{X_{\varepsilon^2 s}^{\delta, \varepsilon}}, \hat{Y}_s^{\delta, \varepsilon}, \mathcal{L}_{\hat{Y}_s^{\delta, \varepsilon}}) d\tilde{W}_s^2 \rangle. \end{aligned} \quad (3.13)$$

由 (1.3) 及 f 和 g 的假设条件可知, 存在 $\tilde{c}_2 > \tilde{c}_1 \geq 0$ 及 $C_0 > 0$, 使得

$$2\langle y, f(\mu, y, \nu) \rangle + \|g(\mu, y, \nu)\|^2 \leq \tilde{c}_1 \mathbb{W}_2(\nu, \delta_0)^2 - \tilde{c}_2 |y|^2 + C_0. \quad (3.14)$$

对 (3.13) 两边同时取期望, 随后对 t 微分, 则有

$$\begin{aligned} d\mathbb{E} |\hat{Y}_t^{\delta, \varepsilon}|^2 &\leq \mathbb{E} [2\langle \hat{Y}_t^{\delta, \varepsilon}, f(\mathcal{L}_{X_{\varepsilon^2 t}^{\delta, \varepsilon}}, \hat{Y}_t^{\delta, \varepsilon}, \mathcal{L}_{\hat{Y}_t^{\delta, \varepsilon}}) \rangle + \|g(\mathcal{L}_{X_{\varepsilon^2 t}^{\delta, \varepsilon}}, \hat{Y}_t^{\delta, \varepsilon}, \mathcal{L}_{\hat{Y}_t^{\delta, \varepsilon}})\|^2] dt \\ &\leq -(\tilde{c}_2 - \tilde{c}_1) \mathbb{E} |\hat{Y}_t^{\delta, \varepsilon}|^2 dt + C_0 dt. \end{aligned}$$

由比较定理可得

$$\mathbb{E} |\hat{Y}_t^{\delta, \varepsilon}|^2 \leq |y|^2 e^{-(\tilde{c}_2 - \tilde{c}_1)t} + C_0 \int_0^t e^{-(\tilde{c}_2 - \tilde{c}_1)(t-s)} ds \leq C(1 + |y|^2), \quad (3.15)$$

其中常数 C 与 t 无关. 回到 (3.13), 将 (3.14) 代入后得到

$$|\hat{Y}_t^{\delta, \varepsilon}|^2 \leq |y|^2 + C_0 t + \tilde{c}_1 \int_0^t \mathbb{E} |\hat{Y}_s^{\delta, \varepsilon}|^2 ds + C \left| \int_0^t \langle \hat{Y}_s^{\delta, \varepsilon}, g(\mathcal{L}_{X_{\varepsilon^2 s}^{\delta, \varepsilon}}, \hat{Y}_s^{\delta, \varepsilon}, \mathcal{L}_{\hat{Y}_s^{\delta, \varepsilon}}) d\tilde{W}_s^2 \rangle \right|.$$

应用 BDG (Burkholder-Davis-Gundy) 不等式和 Young 不等式, 可得

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} |\hat{Y}_t^{\delta, \varepsilon}|^2 \right] &\leq |y|^2 + C_0 T + \tilde{c}_1 \int_0^T \mathbb{E} |\hat{Y}_s^{\delta, \varepsilon}|^2 ds + C \mathbb{E} \left[\int_0^T |\hat{Y}_s^{\delta, \varepsilon}|^2 ds \right]^{\frac{1}{2}} \\ &\leq C + C_0 T + C \int_0^T \mathbb{E} |\hat{Y}_s^{\delta, \varepsilon}|^2 ds \stackrel{(3.15)}{\leq} C_1 (1 + T), \end{aligned}$$

此处 C_1 是不依赖于 T 的常数. 从而有

$$\mathbb{E}\left[\sup_{t \in [0, T]} |Y_t^{\delta, \varepsilon}|^2\right] = \mathbb{E}\left[\sup_{t \in [0, T]} |\hat{Y}_{t/\varepsilon^2}^{\delta, \varepsilon}|^2\right] = \mathbb{E}\left[\sup_{t \in [0, T/\varepsilon^2]} |\hat{Y}_t^{\delta, \varepsilon}|^2\right] \leq C_1 \left(1 + \frac{T}{\varepsilon^2}\right).$$

证毕. $\square$

随后, 我们给出受控系统 (3.11) 中慢过程 $X^{\delta, \varepsilon, h^\delta}$ 的下述估计.

引理 3.5 对任意 $N > 0$, 若 $h^\delta \in \mathcal{A}_N$, 则存在常数 $C_{N, T} > 0$, 使得

$$\mathbb{E}\left[\sup_{t \in [0, T]} |X_t^{\delta, \varepsilon, h^\delta}|^4\right] \leq C_{N, T}(1 + |x|^4 + |y|^4). \quad (3.16)$$

证明 由 Itô 公式可知

$$\begin{aligned} |Y_t^{\delta, \varepsilon, h^\delta}|^2 &= |y|^2 + \frac{2}{\varepsilon^2} \int_0^t \langle Y_s^{\delta, \varepsilon, h^\delta}, f(\mathcal{L}_{X_s^{\delta, \varepsilon}}, Y_s^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{Y_s^{\delta, \varepsilon}}) \rangle ds + \frac{1}{\varepsilon^2} \int_0^t \|g(\mathcal{L}_{X_s^{\delta, \varepsilon}}, Y_s^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{Y_s^{\delta, \varepsilon}})\|^2 ds \\ &\quad + \frac{2}{\sqrt{\delta}\varepsilon} \int_0^t \langle Y_s^{\delta, \varepsilon, h^\delta}, g(\mathcal{L}_{X_s^{\delta, \varepsilon}}, Y_s^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{Y_s^{\delta, \varepsilon}}) P_2 \dot{h}_s^\delta \rangle ds + M_t, \end{aligned} \quad (3.17)$$

其中

$$M_t := \frac{2}{\varepsilon} \int_0^t \langle Y_s^{\delta, \varepsilon, h^\delta}, g(\mathcal{L}_{X_s^{\delta, \varepsilon}}, Y_s^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{Y_s^{\delta, \varepsilon}}) dW_s^2 \rangle.$$

应用 Young 不等式可知

$$\frac{2}{\sqrt{\delta}\varepsilon} \langle Y_s^{\delta, \varepsilon, h^\delta}, g(\mathcal{L}_{X_s^{\delta, \varepsilon}}, Y_s^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{Y_s^{\delta, \varepsilon}}) P_2 \dot{h}_s^\delta \rangle \leq \frac{\tilde{c}_2}{2\varepsilon^2} |Y_s^{\delta, \varepsilon, h^\delta}|^2 + \frac{C}{\delta} |\dot{h}_s^\delta|^2. \quad (3.18)$$

联合 (3.14), (3.17) 和 (3.18), 应用引理 3.3, 有

$$\frac{\tilde{c}_2}{2\varepsilon^2} \int_0^T |Y_s^{\delta, \varepsilon, h^\delta}|^2 ds \leq |y|^2 + \frac{C_T}{\varepsilon^2} (1 + |x|^2 + |y|^2) + \frac{C}{\delta} \int_0^T |\dot{h}_s^\delta|^2 ds + \sup_{t \in [0, T]} |M_t|.$$

将两边平方后取期望, 由 BDG 不等式和 Young 不等式, 可得

$$\begin{aligned} \mathbb{E}\left(\left[\int_0^T |Y_s^{\delta, \varepsilon, h^\delta}|^2 ds\right]^2\right) &\leq C\varepsilon^4 |y|^4 + C_T(1 + |x|^4 + |y|^4) + \frac{C\varepsilon^4}{\delta^2} \mathbb{E}\left(\left[\int_0^T |\dot{h}_s^\delta|^2 ds\right]^2\right) \\ &\quad + C\varepsilon^4 \mathbb{E}\left(\left[\sup_{t \in [0, T]} |M_t|\right]^2\right) \\ &\leq C_T(1 + |x|^4 + |y|^4) + \frac{C_N\varepsilon^4}{\delta^2} + \frac{1}{2} \mathbb{E}\left(\left[\int_0^T |Y_s^{\delta, \varepsilon, h^\delta}|^2 ds\right]^2\right). \end{aligned}$$

由于 $\lim_{\delta \rightarrow 0} \varepsilon^2/\delta = 0$, 不失一般性, 不妨假设 $\varepsilon^2/\delta < 1$, 则

$$\mathbb{E}\left(\left[\int_0^T |Y_s^{\delta, \varepsilon, h^\delta}|^2 ds\right]^2\right) \leq C_{N, T}(1 + |x|^4 + |y|^4).$$

由 Itô 公式可知

$$|X_t^{\delta, \varepsilon, h^\delta}|^2 = |x|^2 + 2 \int_0^t \langle X_s^{\delta, \varepsilon, h^\delta}, b(X_s^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{X_s^{\delta, \varepsilon}}, Y_s^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{Y_s^{\delta, \varepsilon}}) \rangle ds$$

$$\begin{aligned}
& + 2 \int_0^t \langle X_s^{\delta, \varepsilon, h^\delta}, \sigma(X_s^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{X_s^{\delta, \varepsilon}}, \mathcal{L}_{Y_s^{\delta, \varepsilon}}) P_1 \dot{h}_s^\delta \rangle ds + \delta \int_0^t \|\sigma(X_s^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{X_s^{\delta, \varepsilon}}, \mathcal{L}_{Y_s^{\delta, \varepsilon}})\|^2 ds \\
& + 2\sqrt{\delta} \int_0^t \langle X_s^{\delta, \varepsilon, h^\delta}, \sigma(X_s^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{X_s^{\delta, \varepsilon}}, \mathcal{L}_{Y_s^{\delta, \varepsilon}}) dW_s^1 \rangle \\
& =: |x|^2 + \sum_{i=1}^4 I_i^{\delta, \varepsilon}(t).
\end{aligned}$$

由于 b 和 σ 均满足线性增长条件, 联合引理 3.3 可知

$$\begin{aligned}
\mathbb{E} \left[\sup_{t \in [0, T]} |I_1^{\delta, \varepsilon}(t) + I_3^{\delta, \varepsilon}(t)|^2 \right] & \leq C \mathbb{E} \left(\int_0^T (1 + |X_s^{\delta, \varepsilon, h^\delta}|^2 + |Y_s^{\delta, \varepsilon, h^\delta}|^2 + \mathbb{E}|X_s^{\delta, \varepsilon}|^2 + \mathbb{E}|Y_s^{\delta, \varepsilon}|^2) ds \right)^2 \\
& \leq C_T (1 + |x|^4 + |y|^4) + C \int_0^T \mathbb{E}|X_s^{\delta, \varepsilon, h^\delta}|^4 ds \\
& \quad + C \mathbb{E} \left(\left[\int_0^T |Y_s^{\delta, \varepsilon, h^\delta}|^2 ds \right]^2 \right). \tag{3.19}
\end{aligned}$$

由 Hölder 不等式和 Young 不等式及引理 3.3, 可得

$$\begin{aligned}
\mathbb{E} \left[\sup_{t \in [0, T]} |I_2^{\delta, \varepsilon}(t)|^2 \right] & \leq \frac{1}{4} \mathbb{E} \left[\sup_{t \in [0, T]} |X_t^{\delta, \varepsilon, h^\delta}|^4 \right] + C_{N, T} \mathbb{E} \int_0^T \|\sigma(X_s^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{X_s^{\delta, \varepsilon}}, \mathcal{L}_{Y_s^{\delta, \varepsilon}})\|^4 ds \\
& \leq \frac{1}{4} \mathbb{E} \left[\sup_{t \in [0, T]} |X_t^{\delta, \varepsilon, h^\delta}|^4 \right] + C_{N, T} (1 + |x|^4 + |y|^4). \tag{3.20}
\end{aligned}$$

由 BDG 不等式和 Young 不等式及引理 3.3, 即有

$$\begin{aligned}
\mathbb{E} \left[\sup_{t \in [0, T]} |I_4^{\delta, \varepsilon}(t)|^2 \right] & \leq C \mathbb{E} \left(\int_0^T |X_s^{\delta, \varepsilon, h^\delta}|^2 \cdot \|\sigma(X_s^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{X_s^{\delta, \varepsilon}}, \mathcal{L}_{Y_s^{\delta, \varepsilon}})\|^2 ds \right) \\
& \leq \frac{1}{4} \mathbb{E} \left[\sup_{t \in [0, T]} |X_t^{\delta, \varepsilon, h^\delta}|^4 \right] + C_T (1 + |x|^4 + |y|^4). \tag{3.21}
\end{aligned}$$

联合 (3.19)–(3.21), 运用 Gronwall 不等式, 即可证得 (3.16) 成立. $\square$

接下来, 对 $Y_t^{\delta, \varepsilon, h^\delta}$ 与 $Y_t^{\delta, \varepsilon}$ 之间的差值进行估计.

引理 3.6 对任意 $h^\delta \in \mathcal{A}_N$, 存在常数 $C_N > 0$, 使得

$$\int_0^T \mathbb{E} |Y_t^{\delta, \varepsilon, h^\delta} - Y_t^{\delta, \varepsilon}|^2 dt \leq \frac{C_N \varepsilon^2}{\delta}.$$

证明 对 $|Y_t^{\delta, \varepsilon, h^\delta} - Y_t^{\delta, \varepsilon}|^2$ 应用 Itô 公式并取期望, 可得

$$\begin{aligned}
\mathbb{E} |Y_t^{\delta, \varepsilon, h^\delta} - Y_t^{\delta, \varepsilon}|^2 & = \frac{2}{\varepsilon^2} \int_0^t \mathbb{E} \langle Y_s^{\delta, \varepsilon, h^\delta} - Y_s^{\delta, \varepsilon}, f(\mathcal{L}_{X_s^{\delta, \varepsilon}}, Y_s^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{Y_s^{\delta, \varepsilon}}) - f(\mathcal{L}_{X_s^{\delta, \varepsilon}}, Y_s^{\delta, \varepsilon}, \mathcal{L}_{Y_s^{\delta, \varepsilon}}) \rangle ds \\
& \quad + \frac{1}{\varepsilon^2} \int_0^t \mathbb{E} \|g(\mathcal{L}_{X_s^{\delta, \varepsilon}}, Y_s^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{Y_s^{\delta, \varepsilon}}) - g(\mathcal{L}_{X_s^{\delta, \varepsilon}}, Y_s^{\delta, \varepsilon}, \mathcal{L}_{Y_s^{\delta, \varepsilon}})\|^2 ds \\
& \quad + \frac{2}{\sqrt{\delta} \varepsilon} \int_0^t \mathbb{E} \langle Y_s^{\delta, \varepsilon, h^\delta} - Y_s^{\delta, \varepsilon}, g(\mathcal{L}_{X_s^{\delta, \varepsilon}}, Y_s^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{Y_s^{\delta, \varepsilon}}) P_2 \dot{h}_s^\delta \rangle ds.
\end{aligned}$$

结合 (1.3), 应用 Young 不等式, 可得

$$d\mathbb{E} |Y_t^{\delta, \varepsilon, h^\delta} - Y_t^{\delta, \varepsilon}|^2 \leq -\frac{C_2}{\varepsilon^2} \mathbb{E} |Y_t^{\delta, \varepsilon, h^\delta} - Y_t^{\delta, \varepsilon}|^2 dt + \frac{C}{\sqrt{\delta} \varepsilon} \mathbb{E} [|Y_t^{\delta, \varepsilon, h^\delta} - Y_t^{\delta, \varepsilon}| \cdot |\dot{h}_t^\delta|] dt$$

$$\leq -\frac{C_2}{2\varepsilon^2} \mathbb{E}|Y_t^{\delta,\varepsilon,h^\delta} - Y_t^{\delta,\varepsilon}|^2 dt + \frac{C}{\delta} \mathbb{E}|\dot{h}_t^\delta|^2 dt.$$

由比较定理, 可得

$$\mathbb{E}|Y_t^{\delta,\varepsilon,h^\delta} - Y_t^{\delta,\varepsilon}|^2 \leq \frac{C}{\delta} \mathbb{E} \int_0^t e^{-\frac{C_2}{2\varepsilon^2}(t-s)} |\dot{h}_s^\delta|^2 ds.$$

对上式在 $[0, T]$ 上两边同时积分, 利用 Fubini 定理, 有

$$\int_0^T \mathbb{E}|Y_t^{\delta,\varepsilon,h^\delta} - Y_t^{\delta,\varepsilon}|^2 dt \leq \frac{C}{\delta} \mathbb{E} \int_0^T |\dot{h}_s^\delta|^2 \left(\int_s^T e^{-\frac{C_2}{2\varepsilon^2}(t-s)} dt \right) ds \leq C_N \frac{\varepsilon^2}{\delta}.$$

证毕. $\square$

随后, 回顾 Wasserstein 空间中 Poisson 方程理论, 进而在此基础上完成对 (LD_2) 的验证.

考虑带参数 $(x, \mu) \in \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1})$ 的 Poisson 方程

$$-\mathcal{L}_0(\mu, y, \nu)U(x, \mu, y, \nu) = h(x, \mu, y, \nu), \quad (3.22)$$

其中 $\mathcal{L}_0$ 为凝固方程 (1.2) 对应的无穷小生成元

$$\begin{aligned} \mathcal{L}_0(\mu, y, \nu)\varphi(y, \nu) &:= \frac{1}{2} \text{Tr}(gg^*(\mu, y, \nu) \cdot \partial_y^2 \varphi(y, \nu)) + f(\mu, y, \nu) \cdot \partial_y \varphi(y, \nu) \\ &\quad + \int_{\mathbb{R}^{d_2}} \left[\frac{1}{2} \text{Tr}(gg^*(\mu, \tilde{y}, \nu) \cdot \partial_{\tilde{y}} [\partial_\nu \varphi(y, \nu)(\tilde{y})]) \right. \\ &\quad \left. + f(\mu, \tilde{y}, \nu) \cdot \partial_\nu \varphi(y, \nu)(\tilde{y}) \right] \nu(d\tilde{y}) \end{aligned} \quad (3.23)$$

且 h 满足中心化条件, 即

$$\int_{\mathbb{R}^{d_2}} h(x, \mu, y, \zeta^\mu) \zeta^\mu(dy) = 0, \quad \forall (x, \mu) \in \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}). \quad (3.24)$$

现引用文献 [5, 6] 中建立的带参数 Poisson 方程解的正则性结果:

定理 3.1 设 (1.3) 成立且 h 满足中心化条件. 若 $f, g \in C^{(1,1),2,(1,1)}(\mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$, $h \in C^{2,(1,1),2,(1,1)}(\mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$, 则对任意 $(x, \mu) \in \mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1})$, Poisson 方程 (3.22) 存在唯一解 $U(x, \mu, \cdot, \cdot)$ 且

$$U \in C^{2,(1,1),2,(1,1)}(\mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2})).$$

在此基础上, 可建立 $X^{\delta,\varepsilon}$ 与 $\bar{X}^0$ 的差值估计, 其证明过程参照文献 [6, 定理 2.1] 的方法稍作调整即得, 故证明从略.

引理 3.7 对任意 $T > 0$, 存在常数 $C_T > 0$, 使得

$$\sup_{t \in [0, T]} \mathbb{E}|X_t^{\delta,\varepsilon} - \bar{X}_t^0|^2 \leq C_T(\varepsilon^2 + \delta). \quad (3.25)$$

下面证明 (LD_2) 成立.

引理 3.8 对任意 $N > 0$, $\{h^\delta\}_{\delta>0} \subset \mathcal{A}_N$, 当 $\lim_{\delta \rightarrow 0} \varepsilon^2/\delta = 0$ 时, 对任意 $\eta > 0$, 有

$$\lim_{\delta \rightarrow 0} \mathbb{P} \left(\sup_{t \in [0, T]} |X_t^{\delta,\varepsilon,h^\delta} - \bar{X}_t^{h^\delta}| > \eta \right) = 0.$$

证明 基于 (3.11) 和 (3.9), $X_t^{\delta,\varepsilon,h^\delta} - \bar{X}_t^{h^\delta}$ 可作如下分解:

$$\begin{aligned}
X_t^{\delta,\varepsilon,h^\delta} - \bar{X}_t^{h^\delta} &= \int_0^t [b(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) - b(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}})] ds \\
&\quad + \int_0^t [b(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) - \bar{b}(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}})] ds \\
&\quad + \int_0^t [\bar{b}(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}) - \bar{b}(\bar{X}_s^{h^\delta}, \mathcal{L}_{\bar{X}_s^0})] ds + \sqrt{\delta} \int_0^t \sigma(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) dW_s^1 \\
&\quad + \int_0^t [\sigma(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) - \bar{\sigma}(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}})] P_1 \dot{h}_s^\delta ds \\
&\quad + \int_0^t [\bar{\sigma}(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}) - \bar{\sigma}(\bar{X}_s^{h^\delta}, \mathcal{L}_{\bar{X}_s^0})] P_1 \dot{h}_s^\delta ds \\
&=: \sum_{i=1}^6 \mathcal{I}_i^{\delta,\varepsilon}(t).
\end{aligned}$$

由于 b 满足 Lipschitz 条件, 故由引理 3.6 可得

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\mathcal{I}_1^{\delta,\varepsilon}(t)|^2 \right] \leq C \mathbb{E} \int_0^T |Y_s^{\delta,\varepsilon,h^\delta} - Y_s^{\delta,\varepsilon}|^2 ds \leq C_N \frac{\varepsilon^2}{\delta}. \quad (3.26)$$

下面处理 $\mathcal{I}_2^{\delta,\varepsilon}(t)$ 项. 令 $\hat{b}(x, \mu, y, \nu) := b(x, \mu, y, \nu) - \bar{b}(x, \mu)$, 则 $\hat{b} \in C^{2,(1,1),2,(1,1)}(\mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$ 且满足中心化条件. 考虑以下 Poisson 方程:

$$-\mathcal{L}_0(\mu, y, \nu) \Phi(x, \mu, y, \nu) = \hat{b}(x, \mu, y, \nu), \quad (3.27)$$

其中 $\mathcal{L}_0$ 由 (3.23) 定义. 由定理 3.1 可知, 该 (3.27) 存在唯一解 $\Phi \in C^{2,(1,1),2,(1,1)}(\mathbb{R}^{d_1} \times \mathcal{P}_2(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_2} \times \mathcal{P}_2(\mathbb{R}^{d_2}))$. 由 Itô 公式可知

$$\begin{aligned}
&\Phi(X_t^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_t^{\delta,\varepsilon}}, Y_t^{\delta,\varepsilon}, \mathcal{L}_{Y_t^{\delta,\varepsilon}}) \\
&= \Phi(x, \delta_x, y, \delta_y) + \sqrt{\delta} M_t^1 + \frac{1}{\varepsilon} M_t^2 \\
&\quad + \int_0^t b(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) \cdot \partial_x \Phi(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) ds \\
&\quad + \frac{\delta}{2} \int_0^t \text{Tr}(\sigma \sigma^*(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) \cdot \partial_x^2 \Phi(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}})) ds \\
&\quad + \int_0^t \tilde{\mathbb{E}}[b(\tilde{X}_s^{\delta,\varepsilon}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, \tilde{Y}_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) \partial_\mu \Phi(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}})(\tilde{X}_s^{\delta,\varepsilon})] ds \\
&\quad + \frac{\delta}{2} \int_0^t \tilde{\mathbb{E}}[\text{Tr}(\sigma \sigma^*(\tilde{X}_s^{\delta,\varepsilon}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) \partial_{\tilde{x}} [\partial_\mu \Phi(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}})(\tilde{X}_s^{\delta,\varepsilon})])] ds \\
&\quad + \int_0^t \sigma(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) P_1 \dot{h}_s^\delta \cdot \partial_x \Phi(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) ds \\
&\quad + \frac{1}{\varepsilon^2} \int_0^t \mathcal{L}_0(\mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) \Phi(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) ds,
\end{aligned}$$

其中 $(\tilde{X}^{\delta,\varepsilon}, \tilde{Y}^{\delta,\varepsilon})$ 是 $(X^{\delta,\varepsilon}, Y^{\delta,\varepsilon})$ 的独立同分布拷贝, $\tilde{\mathbb{E}}$ 表示对该拷贝的期望且

$$\begin{aligned} M_t^1 &:= \int_0^t \partial_x \Phi(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) \cdot \sigma(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) dW_s^1, \\ M_t^2 &:= \int_0^t \partial_y \Phi(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) \cdot g(\mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) dW_s^2. \end{aligned}$$

由 (3.27) 可知

$$\begin{aligned} & \int_0^t \hat{b}(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) ds \\ &= \varepsilon^2 [\Phi(x, \delta_x, y, \delta_y) - \Phi(X_t^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_t^{\delta,\varepsilon}}, Y_t^{\delta,\varepsilon}, \mathcal{L}_{Y_t^{\delta,\varepsilon}})] \\ &+ [\sqrt{\delta}\varepsilon^2 M_t^1 + \varepsilon M_t^2] + \varepsilon^2 \int_0^t b(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) \cdot \partial_x \Phi(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) ds \\ &+ \frac{\delta\varepsilon^2}{2} \int_0^t \text{Tr}(\sigma\sigma^*(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) \cdot \partial_x^2 \Phi(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}})) ds \\ &+ \varepsilon^2 \int_0^T \tilde{\mathbb{E}}[b(\tilde{X}_s^{\delta,\varepsilon}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, \tilde{Y}_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) \partial_\mu \Phi(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}})(\tilde{X}_s^{\delta,\varepsilon})] ds \\ &+ \frac{\delta\varepsilon^2}{2} \int_0^t \tilde{\mathbb{E}}[\text{Tr}(\sigma\sigma^*(\tilde{X}_s^{\delta,\varepsilon}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) \partial_{\tilde{x}} [\partial_\mu \Phi(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}})(\tilde{X}_s^{\delta,\varepsilon})])] ds \\ &+ \varepsilon^2 \int_0^t \sigma(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) P_1 \dot{h}_s^\delta \cdot \partial_x \Phi(X_s^{\delta,\varepsilon,h^\delta}, \mathcal{L}_{X_s^{\delta,\varepsilon}}, Y_s^{\delta,\varepsilon}, \mathcal{L}_{Y_s^{\delta,\varepsilon}}) ds \\ &=: \sum_{i=1}^7 \mathcal{I}_{2i}^{\delta,\varepsilon}(t). \end{aligned}$$

应用引理 3.4, 可得

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\mathcal{I}_{21}^{\delta,\varepsilon}(t)|^2 \right] \leq C\varepsilon^4 (1 + |y|^2 + \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t^{\delta,\varepsilon}|^2 \right]) \leq C_T \varepsilon^2. \quad (3.28)$$

根据 BDG 不等式及引理 3.3, 可得

$$\mathbb{E} \left[\sup_{t \in [0, T]} |M_t^1|^2 \right] \leq C \mathbb{E} \left(\int_0^T (1 + |Y_s^{\delta,\varepsilon}|^2 + \mathbb{E}|Y_s^{\delta,\varepsilon}|^2)(1 + \mathbb{E}|X_s^{\delta,\varepsilon}|^2 + \mathbb{E}|Y_s^{\delta,\varepsilon}|^2) ds \right) \leq C_T.$$

类似地处理 M_t^2 , 综合可得

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\mathcal{I}_{22}^{\delta,\varepsilon}(t)|^2 \right] \leq C_T \varepsilon^2. \quad (3.29)$$

结合引理 3.3 和 3.5 可知

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} |\mathcal{I}_{23}^{\delta,\varepsilon}(t) + \mathcal{I}_{24}^{\delta,\varepsilon}(t)|^2 \right] \\ & \leq C\varepsilon^4 \mathbb{E} \left(\int_0^T (1 + |X_s^{\delta,\varepsilon,h^\delta}| + |Y_s^{\delta,\varepsilon}| + (\mathbb{E}|X_s^{\delta,\varepsilon}|^2)^{\frac{1}{2}} + (\mathbb{E}|Y_s^{\delta,\varepsilon}|^2)^{\frac{1}{2}})(1 + |Y_s^{\delta,\varepsilon}| + (\mathbb{E}|Y_s^{\delta,\varepsilon}|^2)^{\frac{1}{2}}) ds \right)^2 \end{aligned}$$

$$\begin{aligned}
& + C\delta^2\varepsilon^4\mathbb{E}\left(\int_0^T(1+\mathbb{E}|X_s^{\delta,\varepsilon}|^2+\mathbb{E}|Y_s^{\delta,\varepsilon}|^2)(1+|Y_s^{\delta,\varepsilon}|+(\mathbb{E}|Y_s^{\delta,\varepsilon}|^2)^{\frac{1}{2}})ds\right)^2 \\
& \leq C_T\varepsilon^4\left[\sup_{t\in[0,T]}(1+\mathbb{E}|X_t^{\delta,\varepsilon,h^\delta}|^4+\mathbb{E}|Y_t^{\delta,\varepsilon}|^4)\right] \\
& \leq C_{N,T}\varepsilon^4.
\end{aligned} \tag{3.30}$$

由引理 3.3, 立得

$$\begin{aligned}
& \mathbb{E}\left[\sup_{t\in[0,T]}|\mathcal{I}_{25}^{\delta,\varepsilon}(t)+\mathcal{I}_{26}^{\delta,\varepsilon}(t)|^2\right] \\
& \leq C\varepsilon^4\mathbb{E}\left(\int_0^T(\tilde{\mathbb{E}}|b(\tilde{X}_s^{\delta,\varepsilon},\mathcal{L}_{X_s^{\delta,\varepsilon}},\tilde{Y}_s^{\delta,\varepsilon},\mathcal{L}_{Y_s^{\delta,\varepsilon}})|^2)^{\frac{1}{2}}(\tilde{\mathbb{E}}\|\partial_\mu\Phi(X_s^{\delta,\varepsilon,h^\delta},\mathcal{L}_{X_s^{\delta,\varepsilon}},Y_s^{\delta,\varepsilon},\mathcal{L}_{Y_s^{\delta,\varepsilon}})(\tilde{X}_s^{\delta,\varepsilon})\|^2)^{\frac{1}{2}}ds\right)^2 \\
& \quad + C\delta^2\varepsilon^4\mathbb{E}\left(\int_0^T(\tilde{\mathbb{E}}\|\sigma(\tilde{X}_s^{\delta,\varepsilon},\mathcal{L}_{X_s^{\delta,\varepsilon}},\mathcal{L}_{Y_s^{\delta,\varepsilon}})\|^4)^{\frac{1}{2}}(\tilde{\mathbb{E}}\|\partial_{\tilde{x}}\partial_\mu\Phi(X_s^{\delta,\varepsilon,h^\delta},\mathcal{L}_{X_s^{\delta,\varepsilon}},Y_s^{\delta,\varepsilon},\mathcal{L}_{Y_s^{\delta,\varepsilon}})(\tilde{X}_s^{\delta,\varepsilon})\|^2)^{\frac{1}{2}}ds\right)^2 \\
& \leq C\varepsilon^4\left[\sup_{t\in[0,T]}(1+\mathbb{E}|X_t^{\delta,\varepsilon}|^2+\mathbb{E}|Y_t^{\delta,\varepsilon}|^2)\right]\cdot\mathbb{E}\left(\int_0^T(1+|Y_s^{\delta,\varepsilon}|^2+\mathbb{E}|Y_s^{\delta,\varepsilon}|^2)ds\right) \\
& \quad + C\varepsilon^4\left[\sup_{t\in[0,T]}(1+\mathbb{E}|X_t^{\delta,\varepsilon}|^4+\mathbb{E}|Y_t^{\delta,\varepsilon}|^4)\right]\cdot\mathbb{E}\left(\int_0^T(1+|Y_s^{\delta,\varepsilon}|^2+\mathbb{E}|Y_s^{\delta,\varepsilon}|^2)ds\right) \\
& \leq C_T\varepsilon^4.
\end{aligned} \tag{3.31}$$

类似地,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}|\mathcal{I}_{27}^{\delta,\varepsilon}(t)|^2\right] & \leq \varepsilon^4\mathbb{E}\left(\int_0^T(1+\mathbb{E}|X_s^{\delta,\varepsilon}|^2+\mathbb{E}|Y_s^{\delta,\varepsilon}|^2)(1+|Y_s^{\delta,\varepsilon}|^2+\mathbb{E}|Y_s^{\delta,\varepsilon}|^2)ds\cdot\int_0^T|\dot{h}_s|^2ds\right) \\
& \leq C_{N,T}\varepsilon^4.
\end{aligned} \tag{3.32}$$

综合 (3.28)–(3.32), 可知

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\mathcal{I}_2^{\delta,\varepsilon}(t)|^2\right] \leq C_{N,T}\varepsilon^2. \tag{3.33}$$

由于 $\bar{b}$ 满足 Lipschitz 条件, 所以

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}|\mathcal{I}_3^{\delta,\varepsilon}(t)|^2\right] & \leq C\int_0^T(\mathbb{E}|X_s^{\delta,\varepsilon,h^\delta}-\bar{X}_s^{h^\delta}|^2+\mathbb{E}|X_s^{\delta,\varepsilon}-\bar{X}_s^0|^2)ds \\
& \stackrel{(3.25)}{\leq} C_T(\varepsilon^2+\delta)+C\int_0^T\mathbb{E}|X_s^{\delta,\varepsilon,h^\delta}-\bar{X}_s^{h^\delta}|^2ds.
\end{aligned} \tag{3.34}$$

由 BDG 不等式及引理 3.3, 即得

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\mathcal{I}_4^{\delta,\varepsilon}(t)|^2\right] \leq \delta\mathbb{E}\int_0^T\|\sigma(X_s^{\delta,\varepsilon,h^\delta},\mathcal{L}_{X_s^{\delta,\varepsilon}},\mathcal{L}_{Y_s^{\delta,\varepsilon}})\|^2ds \leq C_T\delta. \tag{3.35}$$

$\mathcal{I}_5^{\delta,\varepsilon}(t)$ 的处理与文献 [6, 定理 2.1] 相似,

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\mathcal{I}_5^{\delta,\varepsilon}(t)|^2\right] \leq C_N\mathbb{E}\left(\int_0^T|\sigma(X_s^{\delta,\varepsilon,h^\delta},\mathcal{L}_{X_s^{\delta,\varepsilon}},\mathcal{L}_{Y_s^{\delta,\varepsilon}})-\bar{\sigma}(X_s^{\delta,\varepsilon,h^\delta},\mathcal{L}_{X_s^{\delta,\varepsilon}})|^2ds\right)$$

$$\begin{aligned}
 &\leq C_N \int_0^T \mathbb{W}_2(\mathcal{L}_{Y_{s/\varepsilon^2}^\mu} | \mu = \mathcal{L}_{X_s^{\delta, \varepsilon}}, \zeta^{\mathcal{L}_{X_s^{\delta, \varepsilon}}})^2 ds + C_N \int_0^T \mathbb{W}_2(\mathcal{L}_{Y_s^{\delta, \varepsilon}}, \mathcal{L}_{Y_{s/\varepsilon^2}^\mu} | \mu = \mathcal{L}_{X_s^{\delta, \varepsilon}})^2 ds \\
 &=: \mathcal{I}_{51}^{\delta, \varepsilon} + \mathcal{I}_{52}^{\delta, \varepsilon},
 \end{aligned}$$

其中 Y^μ 是凝固方程 (1.2) 的唯一强解. 综合文献 [9, 定理 3.1]、[6, 引理 3.5] 和引理 3.3, 可知存在 $\lambda_0 > 0$ 使得

$$\mathcal{I}_{51}^{\delta, \varepsilon} \leq C_N \int_0^T e^{-\frac{2\lambda_0 s}{\varepsilon^2}} \mathbb{W}_2(\delta_y, \zeta^{\mathcal{L}_{X_s^{\delta, \varepsilon}}})^2 ds \leq C_{N,T} \varepsilon^2. \quad (3.36)$$

下面估计 $\mathcal{I}_{52}^{\delta, \varepsilon}$. 令 $\hat{Y}_t^{\delta, \varepsilon} = Y_{\varepsilon^2 t}^{\delta, \varepsilon}$, 其满足方程

$$\hat{Y}_t^{\delta, \varepsilon} = y + \int_0^t f(\mathcal{L}_{X_{\varepsilon^2 s}^{\delta, \varepsilon}}, \hat{Y}_s^{\delta, \varepsilon}, \mathcal{L}_{\hat{Y}_s^{\delta, \varepsilon}}) ds + \int_0^t g(\mathcal{L}_{X_{\varepsilon^2 s}^{\delta, \varepsilon}}, \hat{Y}_s^{\delta, \varepsilon}, \mathcal{L}_{\hat{Y}_s^{\delta, \varepsilon}}) d\tilde{W}_s^2.$$

注意到

$$\mathbb{W}_2(\mathcal{L}_{Y_t^{\delta, \varepsilon}}, \mathcal{L}_{Y_{t/\varepsilon^2}^\mu} | \mu = \mathcal{L}_{X_t^{\delta, \varepsilon}})^2 = \mathbb{W}_2(\mathcal{L}_{\hat{Y}_{t/\varepsilon^2}^{\delta, \varepsilon}}, \mathcal{L}_{Y_{t/\varepsilon^2}^\mu} | \mu = \mathcal{L}_{X_t^{\delta, \varepsilon}})^2 \leq [\mathbb{E}|\hat{Y}_{t/\varepsilon^2}^{\delta, \varepsilon} - Y_{t/\varepsilon^2}^\mu|^2]_{\mu = \mathcal{L}_{X_t^{\delta, \varepsilon}}}. \quad (3.37)$$

由 Itô 公式可知

$$\begin{aligned}
 |\hat{Y}_t^{\delta, \varepsilon} - Y_t^\mu|^2 &= 2 \int_0^t \langle \hat{Y}_s^{\delta, \varepsilon} - Y_s^\mu, f(\mathcal{L}_{X_{\varepsilon^2 s}^{\delta, \varepsilon}}, \hat{Y}_s^{\delta, \varepsilon}, \mathcal{L}_{\hat{Y}_s^{\delta, \varepsilon}}) - f(\mu, Y_s^\mu, \mathcal{L}_{Y_s^\mu}) \rangle ds \\
 &\quad + \int_0^t \|g(\mathcal{L}_{X_{\varepsilon^2 s}^{\delta, \varepsilon}}, \hat{Y}_s^{\delta, \varepsilon}, \mathcal{L}_{\hat{Y}_s^{\delta, \varepsilon}}) - g(\mu, Y_s^\mu, \mathcal{L}_{Y_s^\mu})\|^2 ds \\
 &\quad + 2 \int_0^t \langle \hat{Y}_s^{\delta, \varepsilon} - Y_s^\mu, (g(\mathcal{L}_{X_{\varepsilon^2 s}^{\delta, \varepsilon}}, \hat{Y}_s^{\delta, \varepsilon}, \mathcal{L}_{\hat{Y}_s^{\delta, \varepsilon}}) - g(\mu, Y_s^\mu, \mathcal{L}_{Y_s^\mu})) d\tilde{W}_s^2 \rangle.
 \end{aligned}$$

由 (1.3) 并使用 Young 不等式, 有

$$\mathbb{E}|\hat{Y}_t^{\delta, \varepsilon} - Y_t^\mu|^2 \leq -\frac{c_2 - c_1}{2} \int_0^t \mathbb{E}|\hat{Y}_s^{\delta, \varepsilon} - Y_s^\mu|^2 ds + C \int_0^t \mathbb{W}_2(\mathcal{L}_{X_{\varepsilon^2 s}^{\delta, \varepsilon}}, \mu)^2 ds.$$

由比较定理可知

$$\mathbb{E}|\hat{Y}_t^{\delta, \varepsilon} - Y_t^\mu|^2 \leq C \int_0^t \mathbb{W}_2(\mathcal{L}_{X_{\varepsilon^2 s}^{\delta, \varepsilon}}, \mu)^2 e^{-\frac{c_2 - c_1}{2}(t-s)} ds.$$

由引理 3.3, 不难得到

$$\mathbb{E}|X_t^{\delta, \varepsilon} - X_s^{\delta, \varepsilon}|^2 \leq C_T |t - s|,$$

则

$$\begin{aligned}
 [\mathbb{E}|\hat{Y}_{t/\varepsilon^2}^{\delta, \varepsilon} - Y_{t/\varepsilon^2}^\mu|^2]_{\mu = \mathcal{L}_{X_t^{\delta, \varepsilon}}} &\leq C \int_0^{\frac{t}{\varepsilon^2}} \mathbb{W}_2(\mathcal{L}_{X_{\varepsilon^2 s}^{\delta, \varepsilon}}, \mathcal{L}_{X_t^{\delta, \varepsilon}})^2 e^{-\frac{c_2 - c_1}{2}(\frac{t}{\varepsilon^2} - s)} ds \\
 &\leq \frac{C_T}{\varepsilon^2} \int_0^t (t - s) e^{-\frac{c_2 - c_1}{2} \frac{t-s}{\varepsilon^2}} ds \leq C_T \varepsilon^2,
 \end{aligned}$$

此处的常数 C_T 与 ε 和 t 均无关. 将上式代回 (3.37), 有

$$\mathcal{I}_{52}^{\delta, \varepsilon} \leq C_{N, T} \varepsilon^2. \quad (3.38)$$

结合 (3.36) 和 (3.38), 可得

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\mathcal{I}_5^{\delta, \varepsilon}(t)|^2 \right] \leq C_{N, T} \varepsilon^2. \quad (3.39)$$

由于 $\bar{\sigma}$ 满足 Lipschitz 条件, 所以

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} |\mathcal{I}_6^{\delta, \varepsilon}(t)|^2 \right] &\leq C_N \int_0^T \mathbb{E} |\bar{\sigma}(X_s^{\delta, \varepsilon, h^\delta}, \mathcal{L}_{X_s^{\delta, \varepsilon}}) - \bar{\sigma}(\bar{X}_s^{h^\delta}, \mathcal{L}_{\bar{X}_s^{h^\delta}})|^2 ds \\ &\leq C_N \int_0^T (\mathbb{E} |X_s^{\delta, \varepsilon, h^\delta} - \bar{X}_s^{h^\delta}|^2 + \mathbb{E} |X_s^{\delta, \varepsilon} - \bar{X}_s^0|^2) ds \\ &\stackrel{(3.25)}{\leq} C_{N, T} (\varepsilon^2 + \delta) + C_N \int_0^T \mathbb{E} |X_s^{\delta, \varepsilon, h^\delta} - \bar{X}_s^{h^\delta}|^2 ds. \end{aligned} \quad (3.40)$$

综合 (3.26), (3.33)–(3.35), (3.39) 和 (3.40), 有

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t^{\delta, \varepsilon, h^\delta} - \bar{X}_t^{h^\delta}|^2 \right] \leq C_{N, T} \left(\varepsilon^2 + \delta + \frac{\varepsilon^2}{\delta} \right) + C_N \int_0^T \mathbb{E} |X_s^{\delta, \varepsilon, h^\delta} - \bar{X}_s^{h^\delta}|^2 ds.$$

应用 Gronwall 不等式, 可得

$$\lim_{\delta \rightarrow 0} \mathbb{E} \left[\sup_{t \in [0, T]} |X_t^{\delta, \varepsilon, h^\delta} - \bar{X}_t^{h^\delta}|^2 \right] = 0.$$

由 Chebyshev 不等式可知, 对任意 $\eta > 0$, 有

$$\mathbb{P} \left(\sup_{t \in [0, T]} |X_t^{\delta, \varepsilon, h^\delta} - \bar{X}_t^{h^\delta}| > \eta \right) \leq \frac{\mathbb{E} [\sup_{t \in [0, T]} |X_t^{\delta, \varepsilon, h^\delta} - \bar{X}_t^{h^\delta}|^2]}{\eta^2} \rightarrow 0, \quad \text{当 } \delta \rightarrow 0 \text{ 时.}$$

问题得证. □

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Large deviation principle for slow-fast McKean-Vlasov SDEs

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Abstract In this paper, we prove the Freidlin-Wentzell-type large deviation principle for a general system of fully-coupled slow-fast McKean-Vlasov stochastic dynamics, when the noise intensity $\delta \rightarrow 0$ and the time scale parameter $\varepsilon(\delta)$ satisfies $\varepsilon^2(\delta)/\delta \rightarrow 0$. The coefficients can depend on the distribution of the slow motion and that of the fast motion. The main techniques are based on the Poisson equation and the weak convergence approach for the large deviation principle.

Keywords large deviation principle, slow-fast McKean-Vlasov SDEs, Poisson equation, weak convergence approach

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