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Nevanlinna theory of meromorphic functions on annuli

Dedicated to Professor Yang Lo on the Occasion of his 70th Birthday

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Abstract In this survey paper, we discuss the recent development of Nevanlinna theory of meromorphic functions on annuli, which extends results in Nevanlinna theory in the complex plane or in a disk. In particular, we show that the approach taken on annuli is a unified treatment of functions meromorphic in the complex plane, a disk and an annulus. It allows one to obtain many results in the complex plane and in a disk as corollaries of our results in annuli.

Keywords Nevanlinna theory, meromorphic function, annulus

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1 Introduction

It is well-known that Nevanlinna theory of functions meromorphic in the complex plane and a disk can be extended in a modified form to multiply-connected plane domains. af Hällström [6] in 1940 proved the two main theorems of Nevanlinna for functions meromorphic in multiply-connected domains which admit either a Green's function or a function of the type given by Selberg [17] in the case of a boundary of zero capacity. In general, any proof of such modified theorems on multiply-connected domains, when a Green's function is used, requires that the functions in question are meromorphic on the boundary of the domain. However, it is very often that we have to consider meromorphic functions with singularities on the boundary of a multiply-connected domain.

Since af Hällström's work [6], there are vast references on Nevanlinna theory of meromorphic functions on multiply-connected domains, especially on the punctured complex plane, angular sectors and annuli. For instance, Hayman and Yang [9] in 1982 proved a conjecture of Littlewood on the a-points of a function analytic in an angular sector, under the assumption that the function's lower order is strictly greater than zero. In this survey, we focus on the recent development of the Nevanlinna theory of meromorphic functions on annuli. Historically speaking, the development of the Nevanlinna theory of meromorphic functions on annuli is motivated by its applications to the study of growth properties of a solution of

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complex differential equations, to list a few, as in [2, 3, 5]. Valiron [18] in 1949 also investigated the growth of meromorphic functions on annuli by using a so-called decomposition lemma.

Bieberbach [2] in 1965 systematically and formally defined all Nevanlinna functions (counting, proximity and characteristic functions) and proved many classical theorems in Nevanlinna theory for meromorphic functions on the annuli with the form $\{z : 0 < r_0 \leqslant |z| < \infty\}$. His theorems provide a useful tool for one to study the growth property of meromorphic function near infinity, but, not near the inner circle. Bieberbach proved his version of Jensen's Formula by using a Green's function. Bank and Laine in 1983 pointed out that these results of Bieberbach can be used for meromorphic functions on the annuli with the form $\{z : 0 < |z| < \infty\}$. Their ideas are to write the meromorphic function $f(z)$ on $\{z : 0 < |z| < \infty\}$ as two meromorphic functions on $\{z : 1 \leqslant |z| < \infty\}$. One is f itself with $\{z : 1 \leqslant |z| < \infty\}$, and the other is $f^*(z) = f(1/z)$ with $\{z : 1 \leqslant |z| < \infty\}$. Thus, in order to know the behavior of f near infinity, one should uses f with $\{z : 1 \leqslant |z| < \infty\}$, and to know the behavior of f near the origin, one has to use f^* with $\{z : 1 \leqslant |z| < \infty\}$. Korhonen [13] in 2004 defined all Nevanlinna functions for meromorphic functions on the annuli with the form $\{z : R_1 < |z| < R_2\}$, where $R_1 \geqslant 0$ and $R_2 < \infty$, and extended many classical Nevanlinna theorems under his setting. However, the way he defines the Nevanlinna functions prevents one from knowing the behavior of the function on two boundaries separately, as we may see in Definition B and Theorem B1 below. Around 2006, Kondratyuk, Laine, Khrystiyanyn and Hanyak [12, 10, 11, 7] published a series of work on the Nevanlinna theory of meromorphic functions on multiply-connected domains, such as, annulus and n-punctured complex plane. Among these publications, Kondratyuk and Laine [12] systematically defined all Nevanlinna functions for meromorphic functions on the annuli with the form $\{z : 1/R < |z| < R\}$, where $1 < R \leqslant \infty$. They also rigorously proved a number of classical theorems in Nevanlinna theory for meromorphic functions on $\{z : 1/R < |z| < R\}$. For instance, they proved the Nevanlinna five value theorem for functions meromorphic in $\{z : 0 < |z| < \infty\}$. Lund and Ye [14] in 2009 studied functions meromorphic on the annuli with the form $\{z : R_1 < |z| < R_2\}$, where $R_1 \geqslant 0$ and $R_2 \leqslant \infty$. They defined all Nevanlinna functions with two independent variables. One can be used to approach the inner circle, and the other to the outer circle. Such an approach taken on annuli unifies the treatments of meromorphic functions in the complex plane, a disk and an annulus. Lund [15] also proved many counterparts under this setting for Nevanlinna theory in the complex plan or a disk. These theorems, such as Jensen's formula, logarithmic derivative lemmas, the first and the second main theorems, and Clunie-type lemmas, allow one to obtain many results in the complex plane and in a disk as corollaries of his theorems on annuli.

In this paper, we always assume that the reader is familiar with Nevanlinna theory in the complex plane. Further, in order for the reader to compare all different definitions and theorems for functions meromorphic on different types of annuli, we collect all definitions in Section 2, place similar theorems in each individual section, and post open questions in the last section.

2 Definitions of Nevanlinna functions in $\mathrm{A}(R_1, R_2)$

Let $0 \leqslant R_1 < R_2 \leqslant +\infty$, and write

$$\mathrm{A} = \mathrm{A}(R_1, R_2) = \{z : R_1 < |z| < R_2\}.$$

To denote half-closed annuli we write

$$\mathrm{A}[R_1, R_2) = \{z : R_1 \leqslant |z| < R_2\} \quad \text{and} \quad \mathrm{A}(R_1, R_2] = \{z : R_1 < |z| \leqslant R_2\}.$$

A closed annulus is denoted by $\mathrm{A}[R_1, R_2]$.

We assume without loss of generality that $0 \leqslant R_1 < 1 < R_2 \leqslant +\infty$; otherwise, we make one of the following adjustments. If $0 < R_1 < R_2 < +\infty$, we consider $f(\sqrt{R_1 R_2}z)$, which is meromorphic in $\mathrm{A}(\sqrt{R_1/R_2}, \sqrt{R_2/R_1})$. If $0 = R_1 < R_2 \leqslant 1$ or $1 \leqslant R_1 < R_2 = \infty$, we fix a constant C so that $CR_2 > 1$ in the first case or $CR_2 < 1$ in the second, and consider $f(Cz)$, which will be meromorphic in an annulus containing the unit circle.

Therefore, in the sequel, we always assume that $\mathrm{A} = \mathrm{A}(R_1, R_2)$ with $0 \leqslant R_1 < 1 < R_2 \leqslant +\infty$.

Definition A [2]. *For $\omega \in \cup\{\infty\}$, the counting function of ω for a function f that is meromorphic for $|z| \geqslant \rho_0 > 0$ is given by*

$$N_1(\rho, \omega) = \int_{\rho_0}^{\rho} \frac{n_1(t, \omega)}{t} dt,$$

where $n_1(t, \omega)$ counts the zeros of $f - \omega$ according to multiplicity in $\{z : \rho_0 \leqslant |z| \leqslant t\}$ when $\omega \in \mathrm{C}$ and counts the poles of f in $\{z : \rho_0 \leqslant |z| \leqslant t\}$, otherwise. The proximity function is defined, in the same way as in the plane, by

$$m_1(\rho, \infty) = \int_0^{2\pi} \log^+ |f(\rho e^a)| d\theta.$$

The characteristic function is given by $T_1(\rho) = m_1(\rho, \infty) + N_1(\rho, \infty)$.

In 1983, Bank and Laine [1] considered the functions meromorphic on $\mathrm{A}(0, \infty)$. In order to adopt Bieberbach's notation, they treated such a meromorphic function f on $\mathrm{A}(0, \infty)$ as two meromorphic functions $f(z)$ and $f^*(z) = f(1/z)$ on $\mathrm{A}[1, \infty)$. The corresponding Nevanlinna functions will be denoted by $N_1(r, f)$, $N_1(r, f^*)$, $m_1(r, f)$, $m_1(r, f^*)$, $T_1(r, f)$, and $T_1(r, f^*)$, respectively, in Bieberbach's notation.

Definition B [13]. *Suppose that f is meromorphic in $\mathrm{A}(R_1, R_2)$. Then the counting function of f is defined by*

$$N_{R_1,R_2}(r, \rho, f) = \int_r^{\rho} \frac{n_{R_1,R_2}(t, f)}{t} dt,$$

where $R_1 < r \leqslant \rho < R_2$ and $n_{R_1,R_2}(t, f)$ counts the number of poles in $\mathrm{A}(r, t)$, each pole according to its multiplicity. The proximity function of f is

$$m_{R_1,R_2}(r, \rho, f) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta + \int_0^{2\pi} \log^+ |f(\rho e^{i\theta}))| d\theta,$$

and the characteristic function of f is

$$T_{R_1,R_2}(r, \rho, f) = N_{R_1,R_2}(r, \rho, f) + m_{R_1,R_2}(r, \rho, f),$$

where $R_1 < r \leqslant \rho < R_2$.

Remark. The counting function $N_{R_1,R_2}(r, \rho, f)$ in Definition B seems to have two variables, one is r and the other is ρ. However, from the nature of the definition of $n_{R_1,R_2}(t, f)$, the variable r has to be fixed. Otherwise, $N_{R_1,R_2}(r, \rho, f)$ is not continuous in the first variable r for fixed ρ. Simply put, $N_{R_1,R_2}(r, \rho, f)$ is not well-defined if we treat r and ρ are two independent variables. To see this, consider $f(z) = e^{1/z}(1 - z)^{-1}$ in $\mathrm{A}(0, \infty)$. Fix $\rho > 1$. Then, if $r \leqslant 1$, $n_{R_1,R_2}(t, f) = 0$ when $r < t < 1$ and $n_{R_1,R_2}(t, f) = 1$ when $t \geqslant 1$ and hence

$$N_{R_1,R_2}(1, \rho, f) = \int_1^{\rho} \frac{n_{R_1,R_2}(t, f)}{t} dt = \log \rho.$$

But when $r > 1$, $n_{R_1,R_2}(t, f) = 0$ for any $t > r$ and hence $N_{R_1,R_2}(r, \rho, f) = 0$. Thus the limit as r approaches 1 from the right is 0 but $N_{R_1,R_2}(1, \rho, f) = \log \rho \neq 0$.

Thus, for instance, in order to study the value distribution of f near the circle $\{|z| = R_1\}$, Korhonen has to fix ρ and considers $f(R_1^2/z)$. Consequently, the resulting error term in many theorems in [13] is related to both variables r and ρ, which was not properly expressed. Therefore, generally speaking, the results in [13] are close to those done by Bank and Laine in [1].

Definition C [12]. *Let f be a function meromorphic in $\mathrm{A}(\frac{1}{R}, R)$, where $1 < R \leqslant \infty$. The counting function of f is defined by*

$$N_0(r, f) = \int_1^r \frac{n_0(t, f)}{t} dt,$$

where $n_0(t,f)$ counts the number of poles, according to multiplicity, in $\mathrm{A}(1/t,t)$. *The proximity function $m_0(r,f)$ and the characteristic function $T_0(r,f)$ are defined in a similar manner as done in the complex plane, up to a constant term. They are given by*

$$m_0(r,f)=\int_0^{2\pi}\log^+\left|f\left(\frac{e^{i\theta}}{r}\right)\right|\frac{d\theta}{2\pi}+\int_0^{2\pi}\log^+|f(re^{i\theta})|\frac{d\theta}{2\pi}\ -\int_0^{2\pi}\log^+|f(e^{i\theta})|\frac{d\theta}{\pi},$$

and $T_0(r,f)=m_0(r,f)+N_0(r,f)$, for any $1<r<R$.

Clearly, it is very hard for one to use these definitions to determine the distribution of poles near the inner boundary from that near the outer boundary since all Nevanlinna functions in the definition only have one variable. However, it is quite often that a meromorphic function has infinitely many poles near the inner boundary and at the same time, the function is analytic in the outer circle. In other words, we cannot tell from these definitions that a meromorphic function's growth is faster or slower near each boundary.

Definition D [14]. *Let f be a meromorphic function in* $\mathrm{A}=\mathrm{A}(R_1,R_2)$ *with $0\leqslant R_1<1<R_2\leqslant\infty$. Denote by $n_{\mathrm{A}}(t,f)$ the number of poles counted according to multiplicity in* $\mathrm{A}[t,1)$ *if $t<1$ or in* $\mathrm{A}[1,t]$ *if $t\geqslant 1$. Define the integrated counting function in A by*

$$N_{\mathrm{A}}(r,\rho,f)=\int_r^{\rho}\frac{n_{\mathrm{A}}(t,f)}{t}dt,$$

where $R_1<r\leqslant\rho<R_2$, the proximity function in the annulus by

$$m_{\mathrm{A}}(r,\rho,f)=\int_0^{2\pi}\log^+|f(re^{i\theta})|d\theta+\int_0^{2\pi}\log^+|f(\rho e^{i\theta})|d\theta,$$

and the Nevanlinna characteristic of f in the annulus A *by*

$$T_{\mathrm{A}}(r,\rho,f)=N_{\mathrm{A}}(r,\rho,f)+m_{\mathrm{A}}(r,\rho,f),$$

where $R_1<r\leqslant\rho<R_2$.

Remark. If f is meromorphic in a disk $\mathrm{A}=\mathrm{A}(0,R)$ with $R\leqslant\infty$ and $f(0)\neq 0,\infty$, or without loss of generality, $f(z)\neq 0,\infty$ in $\mathrm{A}(0,1)$, then we have

$$N_{\mathrm{A}}(r,\rho,f)=N(\rho,f)+O(1)\quad\text{and}\quad m_{\mathrm{A}}(r,\rho,f)=m(\rho,f)+O(1);$$

and therefore,

$$T_{\mathrm{A}}(r,\rho,f)=T(\rho,f)+O(1),$$

where N,m and T are the common Nevanlinna functions in a disk or in the complex plane. It follows that Definition D, up to a constant, is the same as those defined in Nevanlinna theory [8, 4].

It is interesting to note from the above definitions that we have to be very careful about defining Nevanlinna's counting function, which essentially reveals the difference between the complex plane case and the annulus case. These definitions later will determine the representations of theorems. In the next several sections, we are going to present corresponding theorems. It is worth pointing out that Bank and Laine [1] considered the growth properties of a meromorphic function f on $\mathrm{A}(0,\infty)$ by investigating meromorphic functions $f(z)$ and $f(1/z)$ on $\mathrm{A}[1,\infty)$. Thus, Bank and Laine pointed out, without giving any details, that the usual results of Nevanlinna theory including Tumura-Clunie type lemma and others remain true by following Bieberbach's theorems. Thus, we are not going to state them as separate theorems in this survey. Further, with the help of a Valiron's factorization theorem, they used their extended Nevanlinna theorems in dealing with meromorphic solutions of complex differential equations.

3 Nevanlinna's first main theorem in $\mathrm{A}(R_1, R_2)$

As usual, Nevanlinna's first main theorem is derived from Jensen's formula. For simplicity, we start with Nevanlinna's first main theorem. Note the notations used in each theorem in following sections are consistent with the corresponding definitions in the previous section.

Theorem A1 [2, p. 99]. *Let f be a non-constant function that is meromorphic in a neighborhood of $\{z : 0 < \rho_0 \leqslant |z| < \infty\}$ and let $\omega \in \mathrm{C}$. Then $m_1(\rho, \omega) + N_1(\rho, \omega) = T_1(\rho) + O(\log \rho)$.*

Since the function is meromorphic on a half-closed annulus and, in particular, $\rho_0 > 0$ is fixed, Bieberbach was able to use a Green's function to prove Jensen's formula, from which Theorem A1 holds. The main difference between Theorem A1 and the first main theorem in the complex plane lies in their error terms. The former is $O(\log \rho)$ and the latter is $O(1)$.

Theorem B1 [13]. *Let f be a meromorphic function in $\mathrm{A}(R_1, R_2)$, and let $a \in \mathrm{C}$. Then*

$$T_{R_1,R_2}\left(r, \rho, \frac{1}{f-a}\right) = T_{R_1,R_2}(r, \rho, f) + O(\log \rho),$$

for all $R_1 < r < \rho < R_2$.

Again, the error term $O(\log \rho)$ should depend on r. This can also be confirmed from his proof or by an example through a little modification of the example we have showed after Definition B. Unfortunately, the error term here does not give any clue how it is related to r. Thus, to make the theorem be correct, one has to fix r. Furthermore, when ρ is fixed and let r approach R_1, the theorem is not true anymore. Therefore, Theorem B1 only describes the growth of f near the outer boundary.

Theorem C1 [12]. *Let f be a meromorphic function in $\mathrm{A}(1/R, R)$ ($R > 0$, or $R = \infty$), not vanishing identically. Then, any fixed $a \in \mathrm{C}$ and for $1 \leqslant r < R$,*

$$T_0\left(r, \frac{1}{f-a}\right) = T_0(r, f) + O(1).$$

Their proof started with an application of the classical argument principle on any annulus contained in $\mathrm{A}(1/R, R)$, then, by using the Fubini theorem, they finished the proof of Jensen's formula. Therefore, Theorem C1 follows. This theorem is almost identical to the first main theorem in the complex plane. Theorem C1 only has one variable and hardly separates the growth property of f near one boundary from the other.

Theorem D1 [14]. *Let $f \not\equiv 0$ or ∞ be a meromorphic function in $\mathrm{A} = \mathrm{A}(R_1, R_2)$ and let $a \in \mathrm{C}$. Then there exists an integer $m = m(f, a)$ such that*

$$\left| T_{\mathrm{A}}\left(r, \rho, \frac{1}{f-a}\right) - T_{\mathrm{A}}(r, \rho, f) + \int_0^{2\pi} \log |f(e^{i\theta}) - a| \frac{d\theta}{\pi} + m \log r\rho \right| \leqslant 2(\log^+ |a| + \log 2),$$

for any r and ρ with $R_1 < r < 1 < \rho < R_2$.

It is clear by setting $r = 1/\rho$ that Theorem D1 becomes Theorem C1. Further, when if $f(0) \neq 0, \infty$, we know from the remark after Definition D that Theorem D1 turns to be, for any $0 < \rho < R_2$,

$$T\left(\rho, \frac{1}{f-a}\right) = T(\rho, f) + O(1),$$

which is a classical Nevanlinna's first main theorem in the complex plan or a disk.

4 Nevanlinna's logarithmic derivative lemma in $\mathrm{A}(R_1, R_2)$

Nevanlinna's logarithmic derivative lemma plays an essential role in Nevanlinna theory. Many important theorems, including the second main theorem, can be proved by utilizing the logarithmic derivative

lemma. In general, Nevanlinna's logarithmic derivative lemma could be proved either by the Poisson-Jensen formula as in [8], or by a differential geometry method as in [16], or by a harmonic analysis method as in [5]. However, when we are dealing with functions meromorphic on annulus, we use a Valiron factorization lemma to prove a logarithmic derivative lemma.

Theorem A2 [2, p. 103]. *Let f be a non-constant function that is meromorphic in a neighborhood of $\{z : 0 < \rho_0 \leqslant |z| < \infty\}$. Then,*

$$m\left(r, \frac{f'}{f}\right) = O\{\max(\log^+ T_1(r), \log r)\},$$

for $r \geqslant \rho_0$ and as $r \to \infty$ outside a set E of finite linear measure.

Essentially, Bieberbach proved the theorem by using a differential geometry method.

Theorem B2 [13]. *Let f be a transcendental meromorphic function in $\mathrm{A}(R_1, R_2)$ and n a positive integer. Then*

$$m_{R_1,R_2}\left(r, \rho, \frac{f^{(n)}}{f}\right) = o(T_{R_1,R_2}(r, \rho, f)),$$

as $r \to R_1$ and $\rho \to R_2$, possibly outside of an exceptional set $E \subset (R_1, R_2)$ for which

$$\max\left\{\int_E \frac{dr}{R_2 - r}, \int_E \frac{dr}{r - R_1}\right\} < \infty$$

if $R_2 < \infty$; and

$$\max\left\{\int_E dr, \int_E \frac{dr}{r - R_1}\right\} < \infty$$

if $R_2 = \infty$.

Again, one may have to understand that the theorem is only true when r is fixed and $\rho \to R_2$.

The following two theorems have a different presentation from Theorems A2 and B2. However, when ones take a proper variable ρ and use a standard growth lemma, the following theorems could have the same presentations as Theorems A2 and B2 have.

Theorem C2 [12]. *Let f be a meromorphic function in $\mathrm{A}(1/R, R)$ $(0 < R \leqslant \infty)$, not vanishing identically. Then, there is a constant $C = C(f)$ such that*

$$m_0\left(r, \frac{f'}{f}\right) \leqslant 4\log^+ \frac{\rho T_0(\rho, f)}{r(\rho - r)} + 4\log^+ \frac{C\rho(1 + \log r)}{r(\rho - r)} + 2(14 + \log r),$$

for $1 \leqslant r < \rho < R$.

The proof of the theorem is based on a Valiron's factorization theorem and a Goldberg-Grishtein's result [5].

Before proceeding next theorem, we must first define some terminology to enable us to discuss a particular class of functions. For functions that are meromorphic in a disk $D(R)$, it is a common practice to define the class of *admissible functions* to be the collection of functions f such that

$$\limsup_{r \to R^-} \frac{T(r, f)}{\log \frac{1}{R - r}} = \infty.$$

We say that a function f that is meromorphic in $\mathrm{A} = \mathrm{A}(R_1, R_2)$ is *left admissible in* A if there exists $\rho_* \in (1, R_2)$ such that

$$\limsup_{r \to R_1^+} \frac{T_{\mathrm{A}}(r, \rho_*, f)}{\log \frac{1}{r - R_1}} = \infty.$$

We say that f is *right admissible in* A when $R_2 < \infty$ if there exists $r_* \in (R_1, 1)$ such that

$$\limsup_{\rho \to R_2^-} \frac{T_{\mathrm{A}}(r_*, \rho, f)}{\log \frac{1}{R_2 - \rho}} = \infty.$$

If $R_2 = \infty$, we replace the above limit with

$$\limsup_{\rho\to\infty} \frac{T_{\mathrm{A}}(r_*, \rho, f)}{\log \rho} = \infty.$$

We say f is *admissible* if f is both right and left admissible.

Theorem D2 [14]. *Let f be a function that is meromorphic and admissible in $\mathrm{A}(R_1, R_2)$ and p and q integers with $p > q \geqslant 0$ and set $s = p - q$. Then there exist $r_0 \in (R_1, 1)$, $\rho_0 \in (1, R_2)$, and constants $C_1 = C_1(p, q, f)$ and $C_2 = C_2(p, q, f)$ such that*

$$m_A\left(r, \rho, \frac{f^{(p)}}{f^{(q)}}\right) \leqslant 2s \log^+ T_{\mathrm{A}}(\hat{r}, \hat{\rho}, f) + s \log^+ G(\hat{r}, r) + s \log^+ \log^+ G(\hat{r}, r) + s \log^+ G(\rho, \hat{\rho}) + s \log^+ \log^+ G(\rho, \hat{\rho}) + O(1),$$

for any $\hat{r}$ and r with $R_1 < \hat{r} < r < r_0$ and any ρ and $\hat{\rho}$ with $\rho_0 < \rho < \hat{\rho} < R_2$, where

$$G(u, v) = \frac{v}{u(v-u)}.$$

Theorem D2, in comparison to Theorems A2, B2 and C2, requires that f is admissible, which makes the characteristic function go to infinity when z approaches the boundaries of $\mathrm{A}(R_1, R_2)$. The proof of Theorem D2 is based on a unified version of logarithmic derivative lemma in [14], which also can be used to derive a classical logarithmic derivative lemma in the complex plane or a disk.

5 Counterparts of other classical theorems in the complex plane

Besides Jensen's formula, the first main theorem and the logarithmic derivative lemma mentioned above, other classical theorems in the complex plane, such as the second main theorem, Clunie-type lemma and the five value theorem, have been extended to an annulus in [2, 13, 12, 14], in their settings. Further, some have showed the applications of these results to complex differential equations. We refer the reader to their original work.

6 A main ingredient in these proofs

The proofs of many above theorems mainly depend on a so-called Valiron's decomposition lemma. The original version of the lemma can be found in [18, p. 15], which is frequently used in the study of the growth property of solutions of a complex differential equations. The basic idea is to use a Valiron's decomposition lemma to write a meromorphic function in $\mathrm{A}(R_1, R_2)$ as a product two meromorphic functions, then apply an inversion transformation to one of these two meromorphic functions to obtain two meromorphic functions in disks. Thus, Nevanlinna theory in the complex plane or a disk can be applied to these meromorphic functions. Of course, it sometimes is quite technical when one puts these two functions together to obtain a final result. Here we just list our version of Valiron's decomposition lemma, which may have some applications beyond Nevanlinna theory.

Valiron's decomposition lemma [14]. *Let $f(z)$ be meromorphic in $\mathrm{A}(R_1, R_2)$, and set $R_1 < R' < R_2$. Then f may be represented as $f(z) = z^m \Phi(z) u(z)$, where*

(a) *the poles and zeros of f in $\mathrm{A}(R_1, R')$ are precisely the poles and zeros of Φ. The poles and zeros of f in $\mathrm{A}[R', R_2)$ are precisely the poles and zeros of u.*

(b) *Φ is meromorphic in $\mathrm{A}(R_1, \infty)$ and analytic and non-zero in $\mathrm{A}[R', \infty]$ and u is meromorphic in $D(R_2)$, and analytic and non-zero in $D(R')$.*

(c) *Φ satisfies*

$$\left|\frac{\Phi'(re^{i\theta})}{\Phi(re^{i\theta})}\right| = O\left(\frac{1}{r^2}\right), \quad \text{as} \quad r \to \infty.$$

(d) *m is an integer.*

7 Open questions

Although many classical Nevanlinna theorems in the complex plane or a disk have been extended to meromorphic functions on annuli, Kondratyuk and Laine [12] have pointed out that it is interesting to investigate the Nevanlinna and the Hardy spaces on a punctured disk, in general. Here we list two questions.

Question 1 [12]**.** Let f be analytic in $\mathrm{A}(0,\infty)$, g entire and $h=g(f)$. Assume that

$$\limsup_{r\to\infty}\frac{\log T_0(r,h)}{\log r}<\infty \quad \text{and} \quad \limsup_{r\to\infty}\frac{\log T_0(r,g)}{\log r}>0,$$

where T_0 is defined in Definition C. Can we write $f(z)=z^pP(z)$, where p is an integer, P is a polynomial?

Motivated by the value distributions of meromorphic fucntions near a singular direction, Zhuan Ye and Jian-Hua Zheng post the following question.

Question 2. Let f be a transcendental meromorphic function in the complex plane C. Is there a sequence of annuli $\{\mathrm{A}_n\}_{n=1}^{\infty}$ such that the Lebegues measure of $\mathrm{C}\setminus(\bigcup_{n=1}^{\infty}\mathrm{A}_n)$ is positive and f takes every value infinitely many times in $\bigcup_{n=1}^{\infty}\mathrm{A}_n$, except two values?

It is known that any transcendental meromorphic function in C takes every value infinitely many times in C, except two values. Question 2 asks if the complex plane C in this statement can be replaced by a sequence of annuli $\{\mathrm{A}_n\}_{n=1}^{\infty}$.

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